RESOLUTION OF SINGULARITIES – SEATTLE LECTURE

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The most influential paper on resolution of singularities is Hironaka's magnum opus [Hir64]. Its starting point is a profound shift in emphasis from resolving singularities of varieties to resolving "singularities of ideal sheaves." Ideal sheaves of smooth or simple normal crossing divisors are the simplest ones. Locally, in a suitable coordinate system, these ideal sheaves are generated by a single monomial. The aim is to transform an arbitrary ideal sheaf into such a "locally monomial" one by a sequence of blow-ups. Ideal sheaves are much more flexible than varieties, and this opens up new ways of running induction.

Since then, resolution of singularities emerged as a very unusual subject whose main object has been a deeper understanding of the proof, rather than the search for new theorems. A better grasp of the proof leads to improved theorems, with the ultimate aim of extending the method to positive characteristic. Two seemingly contradictory aspects make it very interesting to study and develop Hironaka's approach.

First, the method is very robust, in that many variants of the proof work. One can even change basic definitions and be rather confident that the other parts can be modified to fit.

Second, the complexity of the proof is very sensitive to details. Small changes in definitions and presentation may result in major simplifications.

This duality also makes it difficult to write a reasonable historical presentation and to correctly appreciate the contributions of various researchers. Each step ahead can be viewed as small or large, depending on whether we focus on the change in the ideas or on their effect. In some sense, all the results of the past forty years have their seeds in [Hir64], nevertheless, the improvement in the methods has been enormous. Thus, instead of historical notes, here is a list of the most important contributions to the development of the Hironaka method, more or less in historical order: Hironaka [Hir64, Hir77]; Giraud [Gir74]; Villamayor [Vil89, Vil92, Vil96] with his coworkers Bravo [BV01] and Encinas [EV98, EV03]; Bierstone and Milman [BM89, BM91, BM97, BM03]; Encinas and Hauser [EH02] and Włodarczyk [Wło05]. The following proof relies mostly on the works of Villamayor and Włodarczyk.

The methods of Bierstone and Milman and of Encinas and Hauser differ from ours (and from each other) in key technical aspects, though the actual resolution procedures end up very similar.

I have also benefited from the surveys and books [Gir95, Lip75, AHV77, CGO84, HLOQ00, Hau03, Cut04].

Abhyankar's book [Abh66] shows some of the additional formidable difficulties that appear in positive characteristic.

A very elegant approach to resolution following de Jong's results on alterations [dJ96] is developed in the papers [BP96, AdJ97, AW97]. This method produces a resolution as in (2), which is however neither strong (3) nor functorial (4). The version given in [Par99] is especially simple.

Another feature of the study of resolutions is that everyone seems to use different terminology, so I also felt free to introduce my own.

It is very instructive to compare the current methods with Hironaka's "idealistic" paper [Hir77]. The main theme is that resolution becomes simpler if we do not try to control the process very tightly, as illustrated by the following three examples.

(1) The original method of [Hir64] worked with the Hilbert-Samuel function of an ideal sheaf at a point. It was gradually realized that the process simplifies if one considers only the vanishing order of an ideal sheaf—a much cruder invariant.

(2) Two ideals I and J belong to the same *idealistic exponent* if they behave "similarly" with respect to any birational map g. (That is, g^*I and g^*J agree at the generic point of every divisor for every g.) Now we see that it is easier to work with an equivalence relation that requires the "similar" behavior only with respect to some birational maps (namely, composites of smooth blow-ups along subvarieties, where the vanishing order is maximal).

(3) The concept of a *distinguished presentation* attempts to pick a local coordinate system that is optimally adjusted to the resolution of a variety or ideal sheaf. A key result of Włodarczyk [Wło05] says that for a suitably modified ideal, all reasonable choices are equivalent, and thus we do not have to be very careful. Local coordinate systems are not needed at all.

The arguments given here differ from their predecessors in two additional aspects. The first of these is a matter of choice, but the second one makes the structure of the proof patent.

(4) The inductive proof gives resolutions only locally, and patching the local resolutions has been quite difficult. The best way would be to define an invariant on points of varieties $\mathrm{inv}(x, X)$ with values in an ordered set such that

(i) $x \mapsto \mathrm{inv}(x, X)$ is an upper semi continuous function, and

(ii) at each step of the resolution we blow up the locus where the invariant is maximal.

With some modification, this is accomplished in [Vil89, BM97, EH02]. All known invariants are, however, rather complicated. Włodarczyk suggested in [Wło05] that with his methods it should not be necessary to define such an invariant. We show that, by a slight change in the definitions, the resolution algorithm automatically globalizes, obviating the need for the invariant.

(5) Traditionally, the results of Sections 9–12 constituted one intertwined package, which had to be carried through the whole induction together. The introduction of the notions of *D-balanced* and *MC-invariant* ideal sheaves makes it possible to disentangle these to obtain four independent parts.

1. What is a good resolution algorithm?

Before we consider the resolution of singularities in general, it is worthwhile to contemplate what the properties of a good resolution algorithm should be.

Here I concentrate on the case of resolving singularities of varieties only. In practice, one may want to keep track and improve additional objects, for instance, subvarieties or sheaves as well, but for now these variants would only obscure the general picture.

1 (Weakest resolution). *Given a variety X, find a projective variety X' such that X' is smooth and birational to X.*

This is what the Albanese method gives for curves and surfaces. In these cases one can then use this variant to get better resolutions, so we do not lose anything at the end. These stronger forms are, however, not automatic, and it is not at all clear that such a "weakest resolution" would be powerful enough in higher dimensions.

(Note that even if X is not proper, we have to insist on X' being proper; otherwise, one could take the open subset of smooth points of X for X'.)

In practice it is useful, sometimes crucial, to have additional properties.

2 (Resolution). *Given a variety X, find a variety X' and a projective morphism $f : X' \to X$ such that X' is smooth and f is birational.*

This is the usual definition of resolution of singularities.

For many applications this is all one needs, but there are plenty of situations when additional properties would be very useful. Here are some of these.

2.1 (Singularity theory). Let us start with an isolated singularity $x \in X$. One frequently would like to study it by taking a resolution $f : X' \to X$ and connecting the properties of $x \in X$ with properties of the exceptional divisor $E = \mathrm{Ex}(f)$. Here everything works best if E is projective, that is, when $E = f^{-1}(x)$.

It is reasonable to hope that we can achieve this. Indeed, by assumption, $X \setminus \{x\}$ is smooth, so it should be possible to resolve without changing $X \setminus \{x\}$.

2.2 (Open varieties). It is natural to study a noncompact variety X^0 via a compactification $X \supset X^0$. Even if X^0 is smooth, the compactifications that are easy to obtain are usually singular. Then one would like to resolve the singularities of X and get a smooth compactification X'. If we take any resolution $f : X' \to X$, the embedding $X^0 \hookrightarrow X$ does not lift to an embedding $X^0 \hookrightarrow X'$. Thus we would like to find a resolution $f : X' \to X$ such that f is an isomorphism over X^0.

In both of the above examples, we would like the exceptional set E or the boundary $X' \setminus X^0$ to be "simple." Ideally we would like them to be smooth, but this is

rarely possible. The next best situation is when E or $X' \setminus X^0$ are simple normal crossing divisors.

These considerations lead to the following variant.

3 (Strong resolution). *Given a variety X, find a variety X' and a projective morphism $f : X' \to X$ such that we have the following:*

  (1) *X' is smooth and f is birational,*
  (2) *$f : f^{-1}(X^{ns}) \to X^{ns}$ is an isomorphism, and*
  (3) *$f^{-1}(\operatorname{Sing} X)$ is a divisor with simple normal crossings.*

Here $\operatorname{Sing} X$ denotes the set of *singular points* of X and $X^{ns} := X \setminus \operatorname{Sing} X$ the set of *smooth points.*

Strong resolution seems to be the variant that is most frequently used in applications, but sometimes other versions are needed. For instance, one might need condition (3.3) scheme theoretically.

A more important question arises when one has several varieties X_i to work with simultaneously. In this case we may need to know that certain morphisms $\phi_{ij} : X_i \to X_j$ lift to the resolutions $\phi'_{ij} : X'_i \to X'_j$.

It would be nice to have this for all morphisms, which would give a "resolution functor" from the category of all varieties and morphisms to the category of smooth varieties. This is, however, impossible.

Example 3.4. Let $S := (uv - w^2 = 0) \subset \mathbb{A}^3$ be the quadric cone, and consider the morphism

$$\phi : \mathbb{A}^2_{x,y} \to S \quad \text{given by } (x, y) \mapsto (x^2, y^2, xy).$$

The only sensible resolution of $\mathbb{A}^2$ is itself, and any resolution of S dominates the minimal resolution $S' \to S$ obtained by blowing up the origin.

The morphism ϕ lifts to a rational map $\phi' : \mathbb{A}^2 \dashrightarrow S'$, but ϕ' is not a morphism.

It seems that the best one can hope for is that the resolution commutes with smooth morphisms.

4 (Functorial resolution). *For every variety X find a resolution $f_X : X' \to X$ that is functorial with respect to smooth morphisms. That is, any smooth morphism $\phi : X \to Y$ lifts to a smooth morphism $\phi' : X' \to Y'$, which gives a fiber product square*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ \phi'\ } & Y' \\
f_X \downarrow & \square & \downarrow f_Y \\
X & \xrightarrow{\ \phi\ } & Y.
\end{array}
$$

Note that if ϕ' exists, it is unique, and so we indeed get a functor from the category of all varieties and smooth morphisms to the category of smooth varieties and smooth morphisms.

This is quite a strong property with many useful implications.

4.1 (Group actions). Functoriality of resolutions implies that any group action on X lifts to X'. For discrete groups this is just functoriality plus the observation that the only lifting of the identity map on X is the identity map of X'. For an algebraic group G a few more steps are needed; see (9.1).

4.2 (Localization). Let $f_X : X' \to X$ be a functorial resolution. The embedding of any open subset $U \hookrightarrow X$ is smooth, and so the functorial resolution of U is the

restriction of the functorial resolution of X. That is,

$$(f_U : U' \to U) \cong \big(f_X|_{f_X^{-1}(U)} : f_X^{-1}(U) \to U\big).$$

Equivalently, a functorial resolution is Zariski local. More generally, a functorial resolution is étale local since étale morphisms are smooth.

Conversely, we show in (9.2) that any resolution that is functorial with respect to étale morphisms is also functorial with respect to smooth morphisms.

Since any resolution $f : X' \to X$ is birational, it is an isomorphism over some smooth points of X. Any two smooth points of X are étale equivalent, and thus a resolution that is functorial with respect to étale morphisms is an isomorphism over smooth points. Thus any functorial resolution satisfies (3.2).

4.3 (Formal localization). Any sensible étale local construction in algebraic geometry is also formal local. In our case this means that the behavior of the resolution $f_X : X' \to X$ near a point $x \in X$ should depend only on the completion $\widehat{\mathcal{O}}_{x,X}$. (Technically speaking, $\operatorname{Spec} \widehat{\mathcal{O}}_{x,X}$ is not a variety and the map $\operatorname{Spec} \widehat{\mathcal{O}}_{x,X} \to \operatorname{Spec} \mathcal{O}_{x,X}$ is only formally smooth, so this is a stronger condition than functoriality.)

4.4 (Resolution of products). It may appear surprising, but a strong and functorial resolution should *not* commute with products.

For instance, consider the quadric cone $0 \in S = (x^2 + y^2 + z^2 = 0) \subset \mathbb{A}^3$. This is resolved by blowing up the origin $f : S' \to S$ with exceptional curve $C \cong \mathbb{P}^1$. On the other hand,

$$f \times f : S' \times S' \to S \times S$$

cannot be the outcome of an étale local strong resolution. The singular locus of $S \times S$ has two components, $Z_1 = \{0\} \times S$ and $Z_2 = S \times \{0\}$, and correspondingly, the exceptional divisor has two components, $E_1 = C \times S'$ and $E_2 = S' \times C$, which intersect along $C \times C$.

If we work étale locally at $(0,0)$, we cannot tell whether the two branches of the singular locus $Z_1 \cup Z_2$ are on different irreducible components of $\operatorname{Sing} S$ or on one non-normal irreducible component. Correspondingly, the germs of E_1 and E_2 could be on the same irreducible exceptional divisor, and on a strong resolution self-intersections of exceptional divisors are not allowed.

So far we concentrated on the end result $f_X : X' \to X$ of the resolution. Next we look at some properties of the resolution algorithm itself.

5 (Resolution by blowing up smooth centers). *For every variety X find a resolution $f_X : X' \to X$ such that f_X is a composite of morphisms*

$$f_X : X' = X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_1} X_1 \xrightarrow{p_0} X_0 = X,$$

where each $p_i : X_{i+1} \to X_i$ is obtained by blowing up a smooth subvariety $Z_i \subset X_i$.

If we want $f_X : X' \to X$ to be a strong resolution, then the condition $Z_i \subset \operatorname{Sing} X_i$ may also be required, though we need only that $p_0 \cdots p_{i-1}(Z_i) \subset \operatorname{Sing} X$.

Let us note first that in low dimensions some of the best resolution algorithms do not have this property.

(1) The quickest way to resolve a curve is to normalize it. The normalization usually cannot be obtained by blowing up points (though it is a composite of blow-ups of points).

(2) A normal surface can be resolved by repeating the procedure: "blow up the singular points and normalize" [Zar39].

(3) A toric variety is best resolved by toric blow-ups. These are rarely given by blow-ups of subvarieties (cf. [Ful93, 2.6]).

(4) Many of the best-studied singularities are easier to resolve by doing a weighted blow-up first.

(5) The theory of Nash blow-ups offers a—so far mostly hypothetical—approach to resolution that does not rely on blowing up smooth centers; cf. [Hir83].

On the positive side, resolution by blowing up smooth centers has the great advantage that we do not mess up what is already nice. For instance, if we want to resolve X and $Y \supset X$ is a smooth variety containing X, then a resolution by blowing up smooth centers automatically carries along the smooth variety. Thus we get a sequence of smooth varieties Y_i fitting in a diagram

$$
\begin{array}{ccccccc}
X_n & \xrightarrow{p_{n-1}} & X_{n-1} & \cdots & X_1 & \xrightarrow{p_0} & X_0 = X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y_n & \xrightarrow{q_{n-1}} & Y_{n-1} & \cdots & Y_1 & \xrightarrow{q_0} & Y_0 = Y,
\end{array}
$$

where the vertical arrows are closed embeddings.

Once we settle on resolution by successive blowing ups, the main question is how to find the centers that we need to blow up. From the algorithmic point of view, the best outcome would be the following.

6 (Iterative resolution, one blow-up at a time). *For any variety X, identify a subvariety $W(X) \subset X$ consisting of the "worst" singularities. Set $R(X) := B_{W(X)}X$ and $R^m(X) := R(R^{m-1}(X))$ for $m \geq 2$. Then we get resolution by iterating this procedure. That is, $R^m(X)$ is smooth for $m \gg 1$.*

Such an algorithm exists for curves with $W(X) = \operatorname{Sing} X$.

The situation is not so simple in higher dimensions.

Example 6.1. Consider the pinch point, or Whitney umbrella, $S := (x^2 - y^2z = 0) \subset \mathbb{A}^3$. S is singular along the line $(x = y = 0)$. It has a normal crossing point if $z \neq 0$ but a more complicated singularity at $(0,0,0)$.

If we blow up the "worst" singular point $(0,0,0)$ of the surface S, then in the chart with coordinates $x_1 = x/z, y_1 = y/z, z_1 = z$ we get the birational transform $S_1 = (x_1^2 - y_1^2 z_1 = 0)$. This is isomorphic to the original surface.

Thus we conclude that one cannot resolve surfaces by blowing up the "worst" singular point all the time.

We can, however, resolve the pinch point by blowing up the whole singular line. In this case, using the multiplicity (which is a rough invariant) gives the right blow-up, whereas distinguishing the pinch point from a normal crossing point (using some finer invariants) gives the wrong blow-up. The message is that we should not look at the singularities too carefully.

The situation gets even worse for normal 3-folds.

Example 6.2. Consider the 3-fold

$$X := (x^2 + y^2 + z^m t^m = 0) \subset \mathbb{A}^4.$$

The singular locus is the union of the two lines

$$L_1 := (x = y = z = 0) \quad \text{and} \quad L_2 := (x = y = t = 0).$$

There are two reasons why no sensible resolution procedure should start by blowing up either of the lines.

    (i) The two lines are interchanged by the involution $\tau : (x, y, z, t) \mapsto (x, y, t, z)$, and thus they should be blown up in a τ-invariant way.

    (ii) An étale local resolution procedure cannot tell if $L_1 \cup L_2$ is a union of two lines or just two local branches of an irreducible curve. Thus picking one branch does not make sense globally.

Therefore, we must start by blowing up the intersection point $(0, 0, 0, 0)$ (or resort to blowing up a singular subscheme).

Computing the t-chart $x = x_1 t_1, y = y_1 t_1, z = z_1 t_1, t = t_1$, we get

$$X_{1,t} = (x_1^2 + y_1^2 + z_1^m t_1^{2m-2} = 0)$$

and similarly in the z-chart. Thus on $B_0 X$ the singular locus consists of three lines: L_1', L_2' and an exceptional line E.

For $m = 2$ we are thus back to the original situation, and for $m \geq 3$ we made the singularities worse by blowing up. In the $m = 2$ case there is nothing else one can do, and we get our first negative result.

Claim 6.3. There is no iterative resolution algorithm that works one smooth blow-up at a time.

The way out is to notice that our two objections (6.2.i–ii) to first blowing up one of the lines L_1 or L_2 are not so strong when applied to the three lines L_1, L_2 and E on the blow-up $B_0 X$. Indeed, we know that the new exceptional line E is isomorphic to $\mathbb{CP}^1$, and it is invariant under every automorphism lifted from X. Thus we can safely blow up $E \subset B_0 X$ without the risk of running into problems with étale localization. (A key point is that we want to ensure that the process is étale local only on X, not on all the intermediate varieties.) In the $m = 2$ case we can then blow up the birational transforms of the two lines L_1 and L_2 simultaneously, to achieve resolution. (Additional steps are needed for $m \geq 3$.)

In general, we have to ensure that the resolution process has some "memory." That is, at each step the procedure is allowed to use information about the previous blow-ups. For instance, it could keep track of the exceptional divisors that were created by earlier blow-ups of the resolution and in which order they were created.

7 (Other considerations). There are several other ways to judge how good a resolution algorithm is.

7.1 (Elementary methods). A good resolution method should be part of "elementary" algebraic geometry. Both Newton's method of rotating rulers and the Albanese projection method pass this criterion. On the other hand, several of the methods for surfaces rely on more advanced machinery.

7.2 (Computability). In concrete cases, one may wish to explicitly determine resolutions by hand or by a computer. As far as I can tell, the existing methods do rather poorly even on the simplest singularities. In a more theoretical direction, one can ask for the worst case or average complexity of the algorithms. See [BS00b, BS00a, FKP05] for computer implementations.

8. Our resolution is strong and functorial with respect to smooth morphisms, but it is very far from being iterative if we want to work one blow-up at a time. Instead, at each step we specify a long sequence of blow-ups to be performed.

We shift our emphasis from resolution of singularities to principalization of ideal sheaves. While principalization is achieved by a sequence of smooth blow-ups, the resolution of singularities may involve blow-ups of singular centers. Furthermore, at some stage we may blow up a subvariety $Z_i \subset X_i$ along which the variety X_i is smooth. This only happens for subvarieties that sit over the original singular locus, so at the end we still get a strong resolution.

The computability of the algorithm has not been investigated much, but the early indications are not promising. One issue is that starting with, say, a hypersurface $(f = 0) \subset \mathbb{A}^n$ of multiplicity m the first step is to replace the ideal (f) with another ideal $W(f)$, which has more than e^{mn} generators, each of multiplicity at least e^m; see (54.3). Then we reduce to a resolution problem in $(n-1)$-dimensions, and at the next reduction step we again may have an exponential increase of the multiplicity and the number of generators. For any reasonable computer implementation, some shortcuts are essential.

Aside 9. Here we prove the two claims made in (4). These are not used in the rest of the chapter.

Proposition 9.1. The action of an algebraic group G on a scheme X lifts to an action of G on its functorial resolution X'.

Proof. The action of an algebraic group G on a variety X is given by a smooth morphism $m : G \times X \to X$. By functoriality, the resolution $(G \times X)'$ of $G \times X$ is given by the pull-back of X' via m, that is, by $f_X^*(m) : (G \times X)' \to X'$.

On the other hand, the second projection $\pi_2 : G \times X \to X$ is also smooth, and so $(G \times X)' = G \times X'$. Thus we get a commutative diagram

$$
\begin{array}{ccccc}
m' : G \times X' & \cong & (G \times X)' & \xrightarrow{f_X^*(m)} & X' \\
\downarrow id_G \times f_X & & & & \downarrow f_X \\
G \times X & = & G \times X & \xrightarrow{m} & X.
\end{array}
$$

We claim that the composite in the top row $m' : G \times X' \to X'$ defines a group action. This means that the following diagram is commutative, where $m_G : G \times G \to G$ is the group multiplication:

$$
\begin{array}{ccc}
G \times G \times X' & \xrightarrow{id_G \times m'} & G \times X' \\
m_G \times id_{X'} \downarrow & & \downarrow m' \\
G \times X' & \xrightarrow{m'} & X'.
\end{array}
$$

Since $m : G \times X \to X$ defines a group action, we know that the diagram is commutative over a dense open set. Since all schemes in the diagram are separated and reduced, this implies commutativity.   □

Proposition 9.2. Any resolution that is functorial with respect to étale morphisms is also functorial with respect to smooth morphisms.

Proof. As we noted in (4.2), a resolution that is functorial with respect to étale morphisms is an isomorphism over smooth points.

Étale locally, a smooth morphism is a direct product, and so it is sufficient to prove that $(X \times A)' \cong X' \times A$ for any abelian variety A. Such an isomorphism is unique; thus it is enough to prove existence for X proper.

Since $(X \times A)'$ is proper, the connected component of its automorphism group is an algebraic group G (see, for instance, [Kol96, I.1.10]). Let $G_1 \subset G$ denote the subgroup whose elements commute with the projection $\pi : (X \times A)' \to X$.

Let $Z \subset X^{ns}$ be a finite subset. Then $\pi^{-1}(Z) \cong Z \times A$, and the action of A on itself gives a subgroup $j_Z : A \hookrightarrow \operatorname{Aut}(\pi^{-1}(Z))$. There is a natural restriction map $\sigma_Z : G_1 \to \operatorname{Aut}(\pi^{-1}(Z))$; set $G_Z := \sigma_Z^{-1}(j_Z A)$.

As we increase Z, the subgroups G_Z form a decreasing sequence, which eventually stabilizes at a subgroup $G_X \subset G$ such that for every finite set $Z \subset X^{ns}$ the action of G_X on $\pi^{-1}(Z)$ is through the action of A on itself. This gives an injective homomorphism of algebraic groups $G_Z \hookrightarrow A$.

On the other hand, A acts on $X \times A$ by isomorphisms, and by assumption this action lifts to an action of the *discrete* group A on $(X \times A)'$. Thus the injection $G_Z \hookrightarrow A$ has a set-theoretic inverse, so it is an isomorphism of algebraic groups. $\quad \square$

2. Examples of resolutions

We start the study of resolutions with some examples. First, we describe how the resolution method deals with two particular surface singularities $S \subset \mathbb{A}^3$. While these are relatively simple cases, they allow us to isolate six problems facing the method. Four of these we solve later, and we can live with the other two.

Then we see how the problems can be tackled for Weierstrass polynomials and what this solution tells us about the general case. For curves and surfaces, this method was already used in Sections 1.10 and 2.7.

Key idea 10. We look at the trace of $S \subset \mathbb{A}^3$ on a suitable smooth surface $H \subset \mathbb{A}^3$ and reconstruct the whole resolution of S from $S \cap H$.

More precisely, starting with a surface singularity $0 \in S \subset \mathbb{A}^3$ of multiplicity m, we will be guided by $S \cap H$ until the multiplicity of the birational transform of S drops below m. Then we need to repeat the method to achieve further multiplicity reduction.

Example 11 (Resolving $S := (x^2 + y^3 - z^6 = 0) \subset \mathbb{A}^3$)**.** (We already know that the minimal resolution has a single exceptional curve $E \cong (x^2 z + y^3 - z^3 = 0) \subset \mathbb{P}^2$ and it has self-intersection $(E^2) = -1$ but let us forget it for now.)

Set $H := (x = 0) \subset \mathbb{A}^3$, and work with $S \cap H$.

Step 1. Although the trace $S \cap H = (y^3 - z^6 = 0) \subset \mathbb{A}^2$ has multiplicity 3, we came from a multiplicity 2 situation, and we blow up until the multiplicity drops below 2.

Here it takes two blow-ups to achieve this. The crucial local charts and equations are

$$
\begin{aligned}
&x^2 + y^3 - z^6 = 0, \\
&x_1^2 + (y_1^3 - z_1^3)z_1 = 0, \qquad && x_1 = x/z,\, y_1 = y/z,\, z_1 = z, \\
&x_2^2 + (y_2^3 - 1)z_2^2 = 0, \qquad && x_2 = x_1/z_1,\, y_2 = y_1/z_1,\, z_2 = z_1.
\end{aligned}
$$

At this stage the trace of the dual graph of the birational transform of S on the birational transform of H is the following, where the numbers indicate the multiplicity (and not minus the self-intersection number as usual) and $\bullet$ indicates the
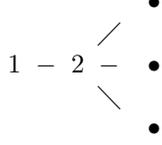
birational transform of the original curve $S \cap H$:

$$
\begin{array}{c}
\bullet \\
\diagup \\
1 \;-\; 2 \;-\; \bullet \\
\diagdown \\
\bullet
\end{array}
$$

Step 2. The birational transform of $S \cap H$ intersects some of the new exceptional curves that appear with positive coefficient. We blow up until these intersections are removed.

In our case each intersection point needs to be blown up twice. At this stage the trace of the birational transform of S on the birational transform of H looks like

$$
\begin{array}{c}
1 \;-\; 0 \;-\; \bullet \\
\diagup \\
1 \;-\; 2 \;-\; 1 \;-\; 0 \;-\; \bullet \\
\diagdown \\
1 \;-\; 0 \;-\; \bullet
\end{array}
$$
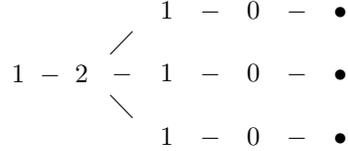
where multiplicity 0 indicates that the curve is no longer contained in the birational transform of H (so strictly speaking, we should not draw it at all).

Step 3. The trace now has multiplicity < 2 along the birational transform of $S \cap H$, but it still has some points of multiplicity ≥ 2. We remove these by blowing up the exceptional curves with multiplicity ≥ 2.

In our case there is only one such curve. After blowing it up, we get the final picture

$$
\begin{array}{c}
1 \;-\; 0 \;-\; \bullet \\
\diagup \\
1 \;-\; \boxed{0} \;-\; 1 \;-\; 0 \;-\; \bullet \\
\diagdown \\
1 \;-\; 0 \;-\; \bullet
\end{array}
$$
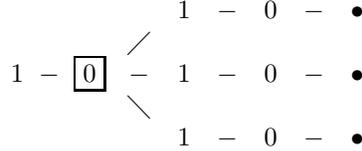
where the boxed curve is elliptic.

More details of the resolution method appear in the following example.

Example 12 (Resolving $S := (x^3 + (y^2 - z^6)^2 + z^{21}) = 0) \subset \mathbb{A}^3$). As before, we look at the trace of S on the plane $H := (x = 0)$ and reconstruct the whole resolution of S from $S \cap H$.

Step 1. Although the trace $S \cap H = ((y^2 - z^6)^2 + z^{21} = 0) \subset \mathbb{A}^2$ has multiplicity 4, we came from a multiplicity 3 situation, and we blow up until the multiplicity drops below 3.

Here it takes three blow-ups to achieve this. The crucial local charts and equations are

$$
\begin{array}{ll}
x^3 + (y^2 - z^6)^2 + z^{21} = 0, & \\
x_1^3 + z_1(y_1^2 - z_1^4)^2 + z_1^{18} = 0, & x_1 = x/z, y_1 = y/z, z_1 = z, \\
x_2^3 + z_2^2(y_2^2 - z_2^2)^2 + z_2^{15} = 0 & x_2 = x_1/z_1, y_2 = y_1/z_1, z_2 = z_1, \\
x_3^3 + z_3^3(y_3^2 - 1)^2 + z_3^{12} = 0, & x_3 = x_2/z_2, y_3 = y_2/z_2, z_3 = z_2.
\end{array}
$$

The birational transform of $S \cap H$ has equation
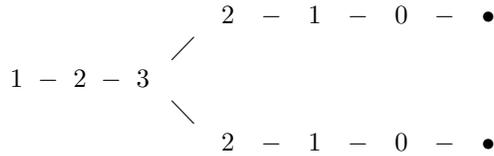
$$(y_3^2 - 1)^2 + z_3^9 = 0$$

and has two higher cusps at $y_3 = \pm 1$ on the last exceptional curve. The trace of the birational transform of S on the birational transform of H looks like

$$
\begin{array}{c}
\quad\bullet \\
\quad / \\
1 \;-\; 2 \;-\; 3 \\
\quad \backslash \\
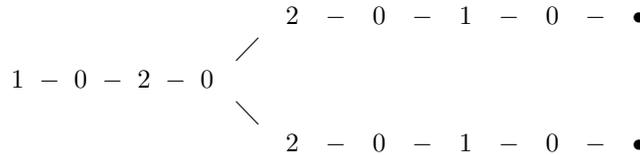\quad\bullet
\end{array}
$$
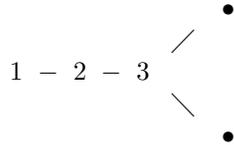
(As before, the numbers indicate the multiplicity, and $\bullet$ indicates the birational transform of the original curve $S \cap H$. Also note that here the curves marked $\bullet$ have multiplicity 2 at their intersection point with the curve marked 3.)

Step 2. The birational transform of $S \cap H$ intersects some of the new exceptional curves that appear with positive coefficient. We blow up until these intersections are removed.

In our case each intersection point needs to be blown up three times, and we get the following picture:

$$
\begin{array}{c}
\quad 2 \;-\; 1 \;-\; 0 \;-\; \bullet \\
\quad / \\
1 \;-\; 2 \;-\; 3 \\
\quad \backslash \\
\quad 2 \;-\; 1 \;-\; 0 \;-\; \bullet
\end{array}
$$

Step 3. The trace now has multiplicity < 3 along the birational transform of $S \cap H$, but it still has some points of multiplicity ≥ 3. There is one exceptional curve with multiplicity ≥ 3; we blow that up. This drops its coefficient from 3 to 0. There are four more points of multiplicity 3, where a curve with multiplicity 2 intersects a curve with multiplicity 1. After blowing these up we get the final picture

$$
\begin{array}{c}
\quad 2 \;-\; 0 \;-\; 1 \;-\; 0 \;-\; \bullet \\
\quad / \\
1 \;-\; 0 \;-\; 2 \;-\; 0 \\
\quad \backslash \\
\quad 2 \;-\; 0 \;-\; 1 \;-\; 0 \;-\; \bullet
\end{array}
$$

13 (Problems with the method). There are at least six different problems with the method. Some are clearly visible from the examples, while some are hidden by the presentation.

Problem 13.1. In (11) we end up with eight exceptional curves, when we need only one to resolve S. In general, for many surfaces the method gives a resolution that is much bigger than the minimal one. However, in higher dimensions there is no minimal resolution, and it is not clear how to measure the "wastefulness" of a resolution.

We will not be able to deal with this issue.

Problem 13.2. The resolution problem for surfaces in $\mathbb{A}^3$ was reduced not to the resolution problem for curves in $\mathbb{A}^2$ but to a related problem that also takes into account exceptional curves and their multiplicities in some way.

We have to set up a somewhat artificial-looking resolution problem that allows true induction on the dimension.

Problem 13.3. The end result of the resolution process guarantees that the birational transform of S has multiplicity < 2 along the birational transform of $H = (x = 0)$, but we have said nothing about the singularities that occur outside the birational transform of H.

There are indeed such singularities if we do not choose H carefully. For instance, if we take $H' := (x - z^2 = 0)$, then at the end of Step.1 of (11), that is, after two blow-ups, the birational transform of H' is $(x_2 - 1 = 0)$, which does not contain the singularity that is at the origin $(x_2 = y_2 = z_2 = 0)$.

Thus a careful choice of H is needed. This is solved by the theory of *maximal contact*, developed by Hironaka and Giraud [Gir74, AHV75].

Problem 13.4. In some cases, the opposite problem happens. All the singularities end up on the birational transforms of H, but we also pick up extra tangencies, so we see too many singularities.

For instance, take $H'' := (x - z^3 = 0)$. Since

$$x^2 + y^3 - z^6 = (x - z^3)(x + z^3) + y^3,$$

the trace of S on H'' is a triple line. The trace shows a 1-dimensional singular set when we have only an isolated singular point.

In other cases, these problems may appear only after many blow-ups.

At first glance, this may not be a problem at all. This simply means that we make some unnecessary blow-ups as well. Indeed, if our aim is to resolve surfaces only, then this problem can be mostly ignored. However, for the general inductive procedure this is a serious difficulty since unnecessary blow-ups can increase the multiplicity. For instance,

$$S = (x^4 + y^2 + yz^2 = 0) \subset \mathbb{A}^3$$

is an isolated double point. If we blow up the line $(x = y = 0)$, in the x-chart we get a triple point

$$x_1^3 + x_1 y_1^2 + y_1 z^2 = 0, \quad \text{where } x = x_1, y = y_1 x_1.$$

One way to solve this problem is to switch from resolving varieties to "resolving" ideal sheaves by introducing a *coefficient ideal* $C(S)$ such that

  (i) resolving S is equivalent to "resolving" $C(S)$, and
  (ii) "resolving" the traces $C(S)|_H$ does not generate extra blow-ups for S.

This change of emphasis is crucial for our approach.

Problem 13.5. No matter how carefully we choose H, we can never end up with a unique choice. For instance, the analytic automorphism of $S = (x^2 + y^3 - z^6 = 0)$,

$$(x, y, z) \mapsto (x + y^3, y \sqrt[3]{1 - 2x - y^3}, z),$$

shows that no internal property distinguishes the choice $x = 0$ from the choice $x + y^3 = 0$.

Even with the careful "maximal contact" choice of H, we end up with cases where the traces $S \cap H$ are not isomorphic. Thus our resolution process seems to depend on the choice of H.

This is again only a minor inconvenience for surfaces, but in higher dimensions we have to deal with patching together the local resolution processes into a global one. (We cannot even avoid this issue by pretending to care only about isolated singularities, since blowing up frequently leads to nonisolated singularities.)

An efficient solution of this problem developed in [Wł0o05] replaces S with an ideal $W(S)$ such that

    (i) resolving S is equivalent to resolving $W(S)$,
   (ii) the traces $W(S)|_H$ are locally isomorphic for all hypersurfaces of maximal contact through $s \in S$ (here "locally" is meant in the analytic or étale topology), and
  (iii) the resolution of $W(S)|_H$ tells us how to resolve $W(S)$.

The local ambiguity is thus removed from the process, and there is no longer a patching problem.

Problem 13.6. At Steps 2 and 3 in (11), the choices we make are not canonical. For instance, in Step 2 we could have blown up the central curve with multiplicity 2 first, to complete the resolution in just one step. Even if we do Step 2 as above, in general there are many curves to blow-up in Step 3, and the order of blow-ups matters. (In $\mathbb{A}^3$, one can blow up two intersecting smooth curves in either order, and the resulting 3-folds are not isomorphic.)

This problem, too, remains unsolved. We make a choice, and it is good enough that the resolutions we get commute with any smooth morphism. Thus we get a resolution that one can call functorial. I would not call it a canonical resolution, since even in the framework of this proof other equally functorial choices are possible.

This is very much connected with the lack of minimal resolutions.

Next we see how Problems (13.2–5) can be approached for hypersurfaces using Weierstrass polynomials. As was the case with curves and surfaces, this example motivates the whole proof. (To be fair, this example provides much better guidance with hindsight. One might argue that the whole history of resolution by smooth blow-ups is but an ever-improving understanding of this single example. It has taken a long time to sort out how to generalize various aspects of it, and it is by no means certain that we have learned all the right lessons.)

Example 14. Let $X \subset \mathbb{C}^{n+1}$ be a hypersurface. Pick a point $0 \in X$, where $\mathrm{mult}_0 X = m$. Choose suitable local coordinates $x_1, \ldots, x_n, z$, and apply the Weierstrass preparation theorem to get (in an analytic neighborhood) an equation of the form

$$z^m + a_1(\mathbf{x})z^{m-1} + \cdots + a_m(\mathbf{x}) = 0$$

for X. We can kill the z^{m-1} term by a substitution $z = y - \frac{1}{m}a_1(\mathbf{x})$ to get another local equation

$$f := y^m + b_2(\mathbf{x})y^{m-2} + \cdots + b_m(\mathbf{x}) = 0. \qquad (14.1)$$

Here $\mathrm{mult}_0 b_i \geq i$ since $\mathrm{mult}_0 X = m$.

Let us blow up the point 0 to get $\pi : B_0 X \to X$, and consider the chart $x_i' = x_i/x_n, x_n' = x_n, y' = y/x_n$. We get an equation for $B_0 X$

$$F := (y')^m + (x_n')^{-2} b_2(x_1' x_n', \ldots, x_n')(y')^{m-2} + \cdots + (x_n')^{-m} b_m(x_1' x_n', \ldots, x_n'). \tag{14.2}$$

Where are the points of multiplicity $\geq m$ on $B_0 X$? Locally we can view $B_0 X$ as a hypersurface in $\mathbb{C}^{n+1}$ given by the equation $F(\mathbf{x}', y') = 0$, and a point p has multiplicity $\geq m$ iff all the $(m-1)$st partials of F vanish. First of all, we get that

$$\frac{\partial^{m-1} F}{\partial y'^{m-1}} = m! \cdot y' \quad \text{vanishes at } p. \tag{14.3}$$

This means that all points of multiplicity $\geq m$ on $B_0 X$ are on the birational transform of the hyperplane $(y = 0)$. Since the new equation (14.2) has the same form as the original (14.1), the conclusion continues to hold after further blow-ups, solving (13.3):

Claim 14.4. After a sequence of blow-ups at points of multiplicity $\geq m$

$$\Pi : X_r = B_{p_{r-1}} X_{r-1} \to X_{r-1} = B_{p_{r-2}} X_{r-2} \to \cdots \to X_1 = B_{p_0} X \to X,$$

all points of multiplicity $\geq m$ on X_r are on the birational transform of the hyperplane $H := (y = 0)$, and all points of X_r have multiplicity $\leq m$.

This property of the hyperplane $(y = 0)$ will be encapsulated by the concept of *hypersurface of maximal contact*.

In order to determine the location of points of multiplicity m, we need to look at all the other $(m-1)$st partials of F restricted to $(y' = 0)$. These can be written as

$$\frac{\partial^{m-1} F}{\partial \mathbf{x}'^{i-1} \partial y'^{m-i}}\big|_{(y'=0)} = (m-i)! \cdot \frac{\partial^{i-1}\big((x_n')^{-i} b_i(x_1' x_n', \ldots, x_n')\big)}{\partial \mathbf{x}'^{i-1}}. \tag{14.5}$$

Thus we can actually read off from $H = (y = 0)$ which points of $B_0 X$ have multiplicity m. For this, however, we need not only the restriction $f|_H = b_m(\mathbf{x})$ but all the other coefficients $b_i(\mathbf{x})$ as well.

There is one further twist. The usual rule for transforming a polynomial under a blow-up is

$$b(x_1, \ldots, x_n) \mapsto (x_n')^{-\operatorname{mult}_0 b} b(x_1' x_n', \ldots, x_n'),$$

but instead we use the rule

$$b_i(x_1, \ldots, x_n) \mapsto (x_n')^{-i} b_i(x_1' x_n', \ldots, x_n').$$

That is, we "pretend" that b_i has multiplicity i at the origin. To handle this, we introduce the notion of a marked function (g, m) and define the birational transform of a marked function (g, m) to be

$$\pi_*^{-1}\big(g(x_1, \ldots, x_n), m\big) := \big((x_n')^{-m} g(x_1' x_n', \ldots, x_n'), m\big). \tag{14.6}$$

Warning. If we change coordinates, the right-hand side of (14.6) changes by a unit. Thus the ideal $(\pi_*^{-1}(g, m))$ is well defined but not $\pi_*^{-1}(g, m)$ itself. Fortunately, this does not lead to any problems.

By induction we define $\Pi_*^{-1}(g, m)$, where Π is a sequence of blow-ups as in (14.4). This leads to a solution of Problems (13.2) and (13.4).

Claim 14.7. After a sequence of blow-ups at points of multiplicity $\geq m$,

$$\Pi : X_r = B_{p_{r-1}} X_{r-1} \to X_{r-1} = B_{p_{r-2}} X_{r-2} \to \cdots \to X_1 = B_{p_0} X \to X,$$

a point $p \in X_r$ has multiplicity $< m$ on X_r iff

   (i) either $p \notin H_r$, the birational transform of H,

   (ii) or there is an index $i = i(p)$ such that

$$\mathrm{mult}_p(\Pi|_{H_r})_*^{-1}\big(b_i(\mathbf{x}), i\big) < i.$$

A further observation is that we can obtain the $b_i(\mathbf{x})$ from the derivatives of f:

$$b_i(\mathbf{x}) = \frac{1}{(m-i)!} \cdot \frac{\partial^{m-i} f}{\partial y^{m-i}}(\mathbf{x}, y)|_H.$$

Thus (14.7) can be restated in a more invariant-looking but also vaguer form.

Principle 14.8. Multiplicity reduction for the $n+1$-variable function $f(\mathbf{x}, y)$ is equivalent to multiplicity reduction for certain n-variable functions constructed from the partial derivatives of f with suitable markings.

14.9. Until now we have completely ignored that everything we do depends on the initial choice of the coordinate system $(x_1, \ldots, x_n, z)$. The fact that in (14.7–8) we get equivalences suggests that the choice of the coordinate system should not matter much. The problem, however, remains: in globalizing the local resolutions constructed above, we have to choose local resolutions out of the many possibilities and hope that the different local choices patch together.

This has been a surprisingly serious obstacle.

3. Statement of the main theorems

So far we have been concentrating on resolution of singularities, but now we switch our focus, and instead of dealing with singular varieties, we consider ideal sheaves on *smooth* varieties. Given an ideal sheaf I on a smooth variety X, our first aim is to write down a birational morphism $g : X' \to X$ such that X' is smooth and the pulled-back ideal sheaf g^*I is locally principal. This is called the principalization of I.

Notation 15. Let $g : Y \to X$ be a morphism of schemes and $I \subset \mathcal{O}_X$ an ideal sheaf. I will be sloppy and use g^*I to denote the *inverse image ideal sheaf* of I. This is the ideal sheaf generated by the pull-backs of local sections of I. (It is denoted by $g^{-1}I \cdot \mathcal{O}_Y$ or by $I \cdot \mathcal{O}_Y$ in [Har77, Sec.II.7].)

We should be mindful that g^*I (as an inverse image ideal sheaf) may differ from the usual sheaf-theoretic pull-back, also commonly denoted by g^*I; see [Har77, II.7.12.2]. This can happen even if X, Y are both smooth.

For the rest of the chapter, we use only inverse image ideal sheaves, so hopefully this should not lead to any confusion.

It is easy to see that resolution of singularities implies principalization. Indeed, let $X_1 := B_I X$ be the blow-up of I with projection $\pi : X_1 \to X$. Then π^*I is locally principal (cf. [Har77, II.7.13]). Thus if $h : X' \to X_1$ is any resolution, then $\pi \circ h : X' \to X$ is a principalization of I.

Our aim, however, is to derive resolution theorems from principalization results. Given a singular variety Z, choose an embedding of Z into a smooth variety X, and let $I_Z \subset \mathcal{O}_X$ be its ideal sheaf. (For Z quasi-projective, we can just take any embedding $Z \hookrightarrow \mathbb{P}^N$ into a projective space, but in general such an embedding may not exist; see (46).) Then we turn a principalization of the ideal sheaf I_Z into a resolution of Z.

In this section we state four, increasingly stronger versions of principalization and derive from them various resolution theorems. The rest of the chapter is then devoted to proving these principalization theorems.

16 (Note on terminology). Principal ideals are much simpler than arbitrary ideals, but they can still be rather complicated since they capture all the intricacies of hypersurface singularities.

An ideal sheaf I on a smooth scheme X is called (locally) *monomial* if the following equivalent conditions hold.
   (1) For every $x \in X$ there are local coordinates z_i and natural numbers c_i such that $I \cdot \mathcal{O}_{x,X} = \prod_i z_i^{c_i} \cdot \mathcal{O}_{x,X}$.
   (2) I is the ideal sheaf of a simple normal crossing divisor (24).

I would like to call a birational morphism $g : X' \to X$, such that X' is smooth and g^*I is monomial, a resolution of I.

However, for many people, the phrase "resolution of an ideal sheaf" brings to mind a long exact sequence

$$\cdots \to E_2 \to E_1 \to I \to 0,$$

where the E_i are locally free sheaves. This has nothing to do with resolution of singularities. Thus, rather reluctantly, I follow convention and talk about *principalization* or *monomialization* of an ideal sheaf I.

We start with the simplest version of principalization (17) and its first consequence, the resolution of indeterminacies of rational maps (18). Then we consider a stronger version of principalization (21), which implies resolution of singularities (22). Monomialization of ideal sheaves is given in (26), which implies strong, functorial resolution for quasi-projective varieties (27). The proof of the strongest variant of monomialization (35) occupies the rest of the chapter. At the end of the section we observe that the functorial properties proved in (35) imply that the monomialization and resolution theorems automatically extend to algebraic and analytic spaces; see (42) and (44).

Theorem 17 (Principalization, I). *Let X be a smooth variety over a field of characteristic zero and $I \subset \mathcal{O}_X$ a nonzero ideal sheaf. Then there is a smooth variety X' and a birational and projective morphism $f : X' \to X$ such that $f^*I \subset \mathcal{O}_{X'}$ is a locally principal ideal sheaf.*

Corollary 18 (Elimination of indeterminacies). *Let X be a smooth variety over a field of characteristic zero and $g : X \dashrightarrow \mathbb{P}$ a rational map to some projective space. Then there is a smooth variety X' and a birational and projective morphism $f : X' \to X$ such that the composite $g \circ f : X' \to \mathbb{P}$ is a morphism.*

Proof. Since $\mathbb{P}$ is projective and X is normal, there is a subset $Z \subset X$ of codimension ≥ 2 such that $g : X \setminus Z \to \mathbb{P}$ is a morphism. Thus $g^*\mathcal{O}_{\mathbb{P}}(1)$ is a line bundle on $X \setminus Z$. Since X is smooth, it extends to a line bundle on X; denote it by L. Let $J \subset L$ be the subsheaf generated by $g^*H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$. Then $I := J \otimes L^{-1}$ is an ideal sheaf, and so by (17) there is a projective morphism $f : X' \to X$ such that $f^*I \subset \mathcal{O}_{X'}$ is a locally principal ideal sheaf.

Thus the global sections

$$(g \circ f)^*H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)\big) \subset H^0(X', f^*L)$$

generate the locally free sheaf $L' := f^*I \otimes f^*L$. Therefore, $g \circ f : X' \to \mathbb{P}$ is a morphism given by the nowhere-vanishing subspace of global sections

$$(g \circ f)^* H^0\big(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)\big) \subset H^0(X', L'). \quad \square$$

Notation 19 (Blow-ups)**.** Let X be a scheme and $Z \subset X$ a closed subscheme. Let $\pi = \pi_{Z,X} : B_Z X \to X$ denote the *blow-up* of Z in X; see [Sha94, II.4] or [Har77, Sec.II.7]. Although resolution by definition involves singular schemes X, we will almost always study the case where X and Z are both smooth, called a *smooth blow-up*. The *exceptional divisor* of a blow-up is $F := \pi_{Z,X}^{-1}(Z) \subset B_Z X$. If $\pi_{Z,X}$ is a smooth blow-up, then F and $B_Z X$ are both smooth.

Warning 20 (Trivial and empty blow-ups)**.** A blow-up is called *trivial* if Z is a Cartier divisor in X. In these cases $\pi_{Z,X} : B_Z X \to X$ is an isomorphism. We also allow the possibility $Z = \emptyset$, called the *empty* blow-up.

We have to deal with trivial blow-ups to make induction work since the blow-up of a codimension 2 smooth subvariety $Z^{n-2} \subset X^n$ corresponds to a trivial blow-up on a smooth hypersurface $Z^{n-2} \subset H^{n-1} \subset X^n$.

Two peculiarities of trivial blow-ups cause trouble.

(1) For a nontrivial smooth blow-up $\pi : B_Z X \to X$, the morphism π determines the center Z, but this fails for a trivial blow-up. One usually thinks of π as *the* blow-up, hiding the dependence on Z. By contrast, we always think of a smooth blow-up as having a specified center.

(2) The exceptional divisor of a trivial blow-up $\pi_{Z,X} : B_Z X \to X$ is $F = Z \subset X$. This is, unfortunately, at variance with the usual definition of exceptional set/divisor (see [Sha94, Sec.II.4.4] or (25)), but it is the right concept for blow-ups.

These are both minor inconveniences, but they could lead to confusion.

Empty blow-ups naturally occur when we restrict a blow-up sequence to an open subset $U \subset X$ and the center of the blow-up is disjoint from U. We will exclude empty blow-ups from the final blow-up sequences, but we have to keep them in mind since they mess up the numbering of the blow-up sequences.

Theorem 21 (Principalization, II)**.** *Let X be a smooth variety over a field of characteristic zero and $I \subset \mathcal{O}_X$ a nonzero ideal sheaf. Then there is a smooth variety X' and a birational and projective morphism $f : X' \to X$ such that*

(1) *$f^*I \subset \mathcal{O}_{X'}$ is a locally principal ideal sheaf,*
(2) *$f : X' \to X$ is an isomorphism over $X \setminus \text{cosupp}\, I$, where $\text{cosupp}\, I$ (or $\text{Supp}(\mathcal{O}_X/I)$) is the cosupport of I, and*
(3) *f is a composite of smooth blow-ups*

$$f : X' = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X.$$

This form of principalization implies resolution of singularities, seemingly by accident. (In practice, one can follow the steps of a principalization method and see how resolution happens, though this is not always easy.)

Corollary 22 (Resolution of singularities, I)**.** *Let X be a quasi-projective variety. Then there is a smooth variety X' and a birational and projective morphism $g : X' \to X$.*

Proof. Choose an embedding of X into a smooth variety P such that $N \geq \dim X + 2$. (For instance, $P = \mathbb{P}^N$ works for all $N \gg \dim X$.) Let $\bar{X} \subset P$ denote the closure and $I \subset \mathcal{O}_P$ its ideal sheaf. Let $\eta_X \in X \subset P$ be the generic point of X.

By (21), there is a sequence of smooth blow-ups

$$\Pi : P' = P_r \xrightarrow{\pi_{r-1}} P_{r-1} \xrightarrow{\pi_{r-2}} \cdots P_1 \xrightarrow{\pi_0} P_0 = P$$

such that $\Pi^* I$ is locally principal.

Since X has codimension ≥ 2, its ideal sheaf I is not locally principal at η_X, and therefore, some blow-up center must contain η_X. Thus, there is a unique j such that $\pi_0 \cdots \pi_{j-1} : P_j \to P$ is a local isomorphism around η_X but $\pi_j : P_{j+1} \to P_j$ is a blow-up with center $Z_j \subset P_j$ such that $\eta_X \in Z_j$.

By (21.2), $\pi_0 \cdots \pi_{j-1}(Z_j) \subset \bar{X}$, and this implies that η_X is the generic point of Z_j. Thus

$$g := \pi_0 \cdots \pi_{j-1} : Z_j \to \bar{X}$$

is birational.

Z_j is smooth since we blow it up, and by (21.3) we only blow up smooth subvarieties. Therefore g is a resolution of singularities of $\bar{X}$. Set $X' := g^{-1}(X) \subset Z_j$. Then $g : X' \to X$ is a resolution of singularities of X. $\qquad\square$

Warning 23. The resolution $g : X' \to X$ constructed in (22) need not be a composite of *smooth* blow-ups. Indeed, the process exhibits g as the composite of blow-ups whose centers are obtained by intersecting the smooth centers Z_i with the birational transforms of X. Such intersections may be singular. See (106) for a concrete example.

We also need a form of resolution that keeps track of a suitable simple normal crossing divisor. This feature is very useful in applications and in the inductive proof.

Definition 24. Let X be a smooth variety and $E = \sum E^i$ a *simple normal crossing divisor* on X. This means that each E^i is smooth, and for each point $x \in X$ one can choose local coordinates $z_1, \ldots, z_n \in m_x$ in the maximal ideal of the local ring $\mathcal{O}_{x,X}$ such that for each i

    (1) either $x \notin E^i$, or
    (2) $E^i = (z_{c(i)} = 0)$ in a neighborhood of x for some $c(i)$, and
    (3) $c(i) \neq c(i')$ if $i \neq i'$.

A subvariety $Z \subset X$ has *simple normal crossings* with E if one can choose $z_1, \ldots, z_n$ as above such that in addition

    (4) $Z = (z_{j_1} = \cdots = z_{j_s} = 0)$ for some $j_1, \ldots, j_s$, again in some open neighborhood of x.

In particular, Z is smooth, and some of the E^i are allowed to contain Z.

If E does not contain Z, then $E|_Z$ is again a simple normal crossing divisor on Z.

Definition 25. Let $g : X' \to X$ be a birational morphism. Its *exceptional set* is the set of points $x' \in X'$ such that g is not a local isomorphism at x'. It is denoted by $\mathrm{Ex}(g)$. If X is smooth, then $\mathrm{Ex}(g)$ is a divisor [Sha94, II.4.4]. Let

$$\Pi : X' = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

be a sequence of smooth blow-ups with centers $Z_i \subset X_i$. Define the *total exceptional set* to be

$$\text{Ex}_{\text{tot}}(\Pi) := \bigcup_{i=0}^{r-1} \left(\pi_i \circ \cdots \circ \pi_{r-1}\right)^{-1}(Z_i).$$

If all the blow-ups are nontrivial, then $\text{Ex}(\Pi) = \text{Ex}_{\text{tot}}(\Pi)$.

Let E be a simple normal crossing divisor on X. We say that the centers Z_i have *simple normal crossings with E* if each blow-up center $Z_i \subset X_i$ has simple normal crossings (24) with

$$(\pi_0 \cdots \pi_{i-1})_*^{-1}(E) + \text{Ex}_{\text{tot}}(\pi_0 \cdots \pi_{i-1}).$$

If this holds, then

$$\Pi_{\text{tot}}^{-1}(E) := \Pi_*^{-1}(E) + \text{Ex}_{\text{tot}}(\Pi)$$

is a simple normal crossing divisor, called the *total transform* of E. (A refinement for divisors with ordered index set will be introduced in (65).)

We can now strengthen the theorem on principalization of ideal sheaves.

Theorem 26 (Principalization, III). *Let X be a smooth variety over a field of characteristic zero, $I \subset \mathcal{O}_X$ a nonzero ideal sheaf and E a simple normal crossing divisor on X. Then there is a sequence of smooth blow-ups*

$$\Pi : R_{I,E}(X) := X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

whose centers have simple normal crossing with E such that

(1) *$\Pi^* I \subset \mathcal{O}_{R_{I,E}(X)}$ is the ideal sheaf of a simple normal crossing divisor, and*
(2) *$\Pi : R_{I,E}(X) \to X$ is functorial on smooth morphisms (4).*

Note that since Π is a composite of smooth blow-ups, $R_{I,E}(X)$ is smooth and $\Pi : R_{I,E}(X) \to X$ is birational and projective.

As a consequence we get strong resolution of singularities for quasi-projective schemes over a field of characteristic zero.

Theorem 27 (Resolution of singularities, II). *Let X be a quasi-projective variety over a field of characteristic zero. Then there is a birational and projective morphism $\Pi : R(X) \to X$ such that*

(1) *$R(X)$ is smooth,*
(2) *$\Pi : R(X) \to X$ is an isomorphism over the smooth locus X^{ns}, and*
(3) *$\Pi^{-1}(\text{Sing } X)$ is a divisor with simple normal crossing.*

Proof. We have already seen in (22) that given a (locally closed) embedding $i : X \hookrightarrow P$ we get a resolution $R(X) \to X$ from the principalization of the ideal sheaf I of the closure of $i(X)$. We need to check that applying (26) to $(P, I, \emptyset)$ gives a strong resolution of X. (We do not claim that $R(X) \to X$ is independent of the embedding $i : X \hookrightarrow P$. This will have to wait until after the stronger principalization theorem (35).)

As in the proof of (22), there is a sequence of smooth blow-ups

$$\Pi : P' = P_r \xrightarrow{\pi_{r-1}} P_{r-1} \xrightarrow{\pi_{r-2}} \cdots P_1 \xrightarrow{\pi_0} P_0 = P$$

such that $\Pi^* I$ is locally principal. Moreover, there is a first blow-up

$$\pi_j : P_{j+1} \to P_j \quad \text{with center } Z_j \subset P_j$$

such that $g := \pi_0 \cdots \pi_{j-1}|_{Z_j} : Z_j \to \bar{X}$ is birational. We claim that $g : Z_j \to \bar{X}$ is a strong resolution of $\bar{X}$, and hence $g : g^{-1}(X) \to X$ is a strong resolution of X.

First, we prove that g is an isomorphism over $\bar{X}^{ns}$. As in (4.2), this follows from the functoriality condition (26.2). Note, however, that (26.2) asserts functoriality for $\Pi : P_r \to P$ but not for the intermediate maps $P_j \to P$. Thus a little extra work is needed. (This seems like a small technical point, but actually it has been the source of serious troubles. The notion of blow-up sequence functors (31) is designed to deal with it.)

Let $F'_{j+1} \subset P_r$ denote the birational transform of $F_{j+1} \subset P_{j+1}$, the exceptional divisor of π_j. Since $F'_{j+1} \subset \mathrm{Ex}_{\mathrm{tot}}(\Pi)$, it is a smooth divisor and so $\Pi|_{F'_{j+1}} : F'_{j+1} \to \bar{X}$ is generically smooth. Thus there is a smooth point $x \in X$ such that $\Pi|_{F'_{j+1}}$ is smooth over x.

For any other smooth point $x' \in X$, the embeddings

$$(x \in X \hookrightarrow P) \quad \text{and} \quad (x' \in X \hookrightarrow P)$$

have isomorphic étale neighborhoods. Thus by (26.2), $\Pi|_{F'_{j+1}}$ is also smooth over x'. We can factor

$$\Pi|_{F'_{j+1}} : F'_{j+1} \to Z_j \xrightarrow{g} \bar{X}.$$

Thus $g : Z_j \to \bar{X}$ is smooth over every smooth point of $\bar{X}$. It is also birational, and thus g is an isomorphism over $\bar{X}^{ns}$.

Since Z_j is smooth, g is not an isomorphism over any point of $\mathrm{Sing}\,\bar{X}$, and thus

$$g^{-1}(\mathrm{Sing}\,\bar{X}) = Z_j \cap \mathrm{Ex}_{\mathrm{tot}}(\pi_0 \cdots \pi_{j-1}),$$

where $\mathrm{Ex}_{\mathrm{tot}}$ denotes the total exceptional divisor (25). Observe that in (26) we can only blow-up Z_j if it has simple normal crossings with $\mathrm{Ex}_{\mathrm{tot}}(\pi_0 \cdots \pi_{j-1})$; hence

$$g^{-1}(\mathrm{Sing}\,\bar{X}) = Z_j \cap \mathrm{Ex}_{\mathrm{tot}}(\pi_0 \cdots \pi_{j-1})$$

is a simple normal crossing divisor on Z_j. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 28. The proof of the implication (26) $\Rightarrow$ (27) also works for any scheme that can be embedded into a smooth variety. We see in (46) that not all schemes can be embedded into a smooth scheme, so in general one has to proceed differently. It is worthwhile to contemplate further the local nature of resolutions and its consequences.

Let X be a scheme of finite type and $X = \cup U_i$ an affine cover. For each U_i (27) gives a resolution $R(U_i) \to U_i$, and we would like to patch these together to $R(X) \to X$.

First, we need to show that $R(U_i)$ is well defined; that is, it does not depend on the embedding $i : U_i \hookrightarrow P$ chosen in the proof of (27).

Second, we need to show that $R(U_i)$ and $R(U_j)$ agree over the intersection $U_i \cap U_j$.

If these hold, then the $R(U_i)$ patch together into a resolution $R(X) \to X$, but there is one problem. $R(X) \to X$ is *locally* projective, but it may not be *globally* projective. The following is an example of this type.

Example 28.1. Let X be a smooth 3-fold and C_1, C_2 a pair of irreducible curves, intersecting at two points p_1, p_2. Assume, furthermore, that C_i is smooth away from p_i, where it has a cusp whose tangent plane is transversal to the other curve. Let $I \subset \mathcal{O}_X$ be the ideal sheaf of $C_1 \cup C_2$.

On $U_1 = X \setminus \{p_1\}$, the curve C_1 is smooth; we can blow it up first. The birational transform of C_2 becomes smooth, and we can blow it up next to get $Y_1 \to U_1$. Over

$U_2 = X \setminus \{p_2\}$ we would work in the other order. Over $U_1 \cap U_2$ we get the same thing, and thus Y_1 and Y_2 glue together to a variety Y such that $Y \to X$ is proper and locally projective but not globally projective.

We see that the gluing problem comes from the circumstance that the birational map $Y_1 \cap Y_2 \to U_1 \cap U_2$ is the blow-up of two disjoint curves, and we do not know which one to blow up first.

For a sensible resolution algorithm there is only one choice: we have to blow them up at the same time. Thus in the above example, the "correct" method is to blow up the points p_1, p_2 first. The curves C_1, C_2 become smooth and disjoint, and then both can be blown up. (More blow-ups are needed if we want to have only simple normal crossings.)

These problems can be avoided if we make (26) sharper. A key point is to prove functoriality conditions not only for the end result $R_{I,E}(X)$ but for all intermediate steps, including the center of each blow-up.

Definition 29 (Blow-up sequences). Let X be a scheme. A *blow-up sequence* of length r starting with X is a chain of morphisms

$$\Pi : X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X, \tag{29.1}$$
$$\cup \qquad\qquad \cup \qquad \cup$$
$$Z_{r-1} \quad \cdots \quad Z_1 \qquad Z_0$$

where each $\pi_i = \pi_{Z_i, X_i} : X_{i+1} \to X_i$ is a blow-up with center $Z_i \subset X_i$ and exceptional divisor $F_{i+1} \subset X_{i+1}$. Set

$$\Pi_{ij} := \pi_j \circ \cdots \circ \pi_{i-1} : X_i \to X_j \quad \text{and} \quad \Pi_i := \Pi_{i0} : X_i \to X_0.$$

We say that (29.1) is a *smooth blow-up sequence* if each $\pi_i : X_{i+1} \to X_i$ is a smooth blow-up.

We allow trivial and empty blow ups (20).

For the rest of the chapter, π always denotes a blow-up, Π_{ij} a composite of blow-ups and Π the composite of all blow-ups in a blow-up sequence (whose length we frequently leave unspecified). We usually drop the centers Z_i from the notation, to avoid cluttering up the diagrams.

Definition 30 (Transforming blow-up sequences). There are three basic ways to transform blow-up sequences from one scheme to another. Let $\mathbf{B} :=$

$$\Pi : X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$
$$\cup \qquad\qquad \cup \qquad \cup$$
$$Z_{r-1} \quad \cdots \quad Z_1 \qquad Z_0$$

be a blow-up sequence starting with X.

30.1. For a smooth morphism $h : Y \to X$ define the *pull-back* $h^*\mathbf{B}$ to be the blow-up sequence

$$h^*\Pi : X_r \times_X Y \xrightarrow{h^*\pi_{r-1}} X_{r-1} \times_X Y \quad \cdots \quad X_1 \times_X Y \xrightarrow{h^*\pi_0} X_0 \times_X Y = Y.$$
$$\cup \qquad\qquad \cup \qquad \cup$$
$$Z_{r-1} \times_X Y \quad \cdots \quad Z_1 \times_X Y \qquad Z_0 \times_X Y$$

If $\mathbf{B}$ is a smooth blow-up sequence then so is $h^*\mathbf{B}$. If h is surjective then $h^*\mathbf{B}$ determines $\mathbf{B}$ uniquely. However, if h is not surjective, then $h^*\mathbf{B}$ may contain some empty blow-ups, and we lose information about the centers living above $X \setminus h(Y)$.

30.2. Let X be a scheme and $j : S \hookrightarrow X$ a closed subscheme. Given a blow-up sequence $\mathbf{B}$ starting with X as above, define its *restriction* to S as the sequence

$$\Pi^S : S_r \xrightarrow{\pi^S_{r-1}} S_{r-1} \xrightarrow{\pi^S_{r-2}} \cdots \xrightarrow{\pi^S_1} S_1 \xrightarrow{\pi^S_0} S_0 = S.$$
$$\cup \qquad\qquad\qquad \cup \qquad\qquad \cup$$
$$Z_{r-1} \cap S_{r-1} \qquad \cdots \qquad Z_1 \cap S_1 \qquad Z_0 \cap S_0$$

It is denoted by $j^*\mathbf{B}$ or $\mathbf{B}|_S$.

Note that $S_{i+1} := B_{Z_i \cap S_i} S_i$ is naturally identified with the birational transform $(\pi_i)_*^{-1} S_i \subset X_{i+1}$ (cf. [Har77, II.7.15]), thus there are natural embeddings $S_i \hookrightarrow X_i$ for every i.

The restriction of a smooth blow-up sequence need not be a smooth blow-up sequence.

30.3. Conversely, let $\mathbf{B}(S) :=$

$$\Pi := S_r \xrightarrow{\pi_{r-1}} S_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} S_1 \xrightarrow{\pi_0} S_0 = S$$

be a blow-up sequence with centers $Z_i^S \subset S_i$. Define its *push-forward* as the sequence $j_*\mathbf{B} :=$

$$\Pi^X : X_r \xrightarrow{\pi^X_{r-1}} X_{r-1} \xrightarrow{\pi^X_{r-2}} \cdots \xrightarrow{\pi^X_1} X_1 \xrightarrow{\pi^X_0} X_0 = X,$$

whose centers $Z_i^X \subset X_i$ are defined inductively as $Z_i^X := (j_i)_* Z_i^S$, where the $j_i : S_i \hookrightarrow X_i$ are the natural inclusions. Thus, for all practical purposes, $Z_i^X = Z_i^S$.

If $\mathbf{B}$ is a smooth blow-up sequence, then so is $j_*\mathbf{B}$.

Definition 31 (Blow-up sequence functors). A *blow-up sequence functor* is a functor $\mathcal{B}$ whose

(1) inputs are triples (X, I, E), where X is a scheme, $I \subset \mathcal{O}_X$ an ideal sheaf that is nonzero on every irreducible component and E a divisor on X with ordered index set, and

(2) outputs are blow-up sequences

$$\Pi : X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$
$$\cup \qquad\qquad\qquad \cup \qquad \cup$$
$$Z_{r-1} \qquad \cdots \qquad Z_1 \qquad Z_0$$

with specified centers. Here the length of the sequence r, the schemes X_i and the centers Z_i all depend on (X, I, E). (Later we will add ideal sheaves I_i and divisors E_i to the notation.)

If each Z_i is smooth, then a nontrivial blow-up $\pi_i : X_{i+1} \to X_i$ uniquely determines Z_i, so we can drop Z_i from the notation. However, in general many different centers give the same birational map.

The (partial) *resolution functor* $\mathcal{R}$ *associated* to a blow-up sequence functor $\mathcal{B}$ is the functor that sends (X, I, E) to the end result of the blow-up sequence

$$\mathcal{R} : (X, I, E) \mapsto (\Pi : X_r \to X).$$

Sometimes we write simply $\mathcal{R}_{(I,E)}(X) = X_r$.

32 (Empty blow-up convention). We basically try to avoid empty blow-ups, but we are forced to deal with them because a pull-back or a restriction of a nonempty blow-up may be an empty blow-up.

Instead of saying repeatedly that we perform a certain blow-up unless its center is empty, we adopt the convention that the final outputs of the named blow-up sequence functors $\mathcal{BD}, \mathcal{BMO}, \mathcal{BO}, \mathcal{BP}$ do not contain empty blow-ups.

The process of their construction may contain blow-ups that are empty in certain cases. (For instance, we may be told to blow up $E^1 \cap E^2$ and the intersection may be empty.) These steps are then ignored without explicit mention whenever they happen to lead to empty blow-ups.

Remark 33. The end result of a sequence of blow-ups $\Pi : X_r \to X$ often determines the whole sequence, but this is not always the case.

First, there are some genuine counterexamples. Let $p \in C$ be a smooth pointed curve in a smooth 3-fold X_0. We can first blow up p and then the birational transform of C to get

$$\Pi : X_2 \xrightarrow{\pi_1} X_1 = B_p X_0 \xrightarrow{\pi_0} X_0,$$

with exceptional divisors $E_0, E_1 \subset X_2$, or we can blow up first C and then the preimage $D = \sigma_0^{-1}(p)$ to get

$$\Sigma : X_2' \xrightarrow{\sigma_1} X_1' = B_C X_0 \xrightarrow{\sigma_0} X_0$$

with exceptional divisors $E_0', E_1' \subset X_2'$.

It is easy to see that $X_2 \cong X_2'$, and under this isomorphism E_1 corresponds to E_0' and E_0 corresponds to E_1'.

Second, there are some "silly" counterexamples. If $Z_1, Z_2 \subset X$ are two disjoint smooth subvarieties, then we get the same result whether we blow up first Z_1 and then Z_2, or first Z_2 and then Z_1, or in one step we blow up $Z_1 \cup Z_2$.

While it seems downright stupid to distinguish between these three processes, it is precisely this ambiguity that caused the difficulties in (28.1).

It is also convenient to have a unified way to look at the functoriality properties of various resolutions.

34 (Functoriality package)**.** There are three functoriality properties of blow-up sequence functors $\mathcal{B}$ that we are interested in. Note that in all three cases the claimed isomorphism is unique, and hence the existence is a local question.

Functoriality for étale morphisms is an essential ingredient of the proof. As noted in (9.2), this is equivalent to functoriality for smooth morphisms (34.1). Independence of the base field (34.2) is very useful in applications, but it is not needed for the proofs.

Functoriality for closed embeddings (34.3) is used for resolution of singularities, but it is not needed for the principalization theorems. This property is quite delicate, and we are not able to prove it in full generality, see (71).

34.1 (Smooth morphisms). We would like our resolutions to commute with smooth morphisms, and it is best to build this into the blow-up sequence functors.

We say that a blow-up sequence functor $\mathcal{B}$ commutes with h if

$$\mathcal{B}\big(Y, h^* I, h^{-1}(E)\big) = h^* \mathcal{B}(X, I, E).$$

This sounds quite reasonable until one notices that even when $Y \to X$ is an open immersion it can happen that $Z_0 \times_X Y$ is empty. It is, however, reasonable to expect that a good blow-up sequence functor commutes with smooth *surjections*.

Therefore, we say that $\mathcal{B}$ commutes with smooth morphisms if

- $\mathcal{B}$ commutes with every smooth surjection h, and
- for every smooth morphism h, $\mathcal{B}(Y, h^*I, h^{-1}(E))$ is obtained from the pull-back $h^*\mathcal{B}(X, I, E)$ by deleting every blow-up $h^*\pi_i$ whose center is empty and reindexing the resulting blow-up sequence.

34.2 (Change of fields). We also would like the resolution to be independent of the field we work with.

Let $\sigma : K \hookrightarrow L$ be a field extension. Given a K-scheme of finite type $X_K \to \operatorname{Spec} K$, we can view $\operatorname{Spec} L$ as a scheme over $\operatorname{Spec} K$ (possibly not of finite type) and take the fiber product

$$X_{L,\sigma} := X_K \times_{\operatorname{Spec} K} \operatorname{Spec} L,$$

which is an L-scheme of finite type. If I is an ideal sheaf and E a divisor on X, then similarly we get $I_{L,\sigma}$ and $E_{L,\sigma}$.

We say that $\mathcal{B}$ *commutes* with σ if $\mathcal{B}(X_{L,\sigma}, I_{L,\sigma}, E_{L,\sigma})$ is the blow-up sequence

$$\Pi_{L,\sigma} : (X_r)_{L,\sigma} \quad \overset{(\pi_{r-1})_{L,\sigma}}{\longrightarrow} \quad (X_{r-1})_{L,\sigma} \quad \cdots \quad (X_1)_{L,\sigma} \quad \overset{(\pi_0)_{L,\sigma}}{\longrightarrow} \quad (X_0)_{L,\sigma}.$$
$$\cup \qquad\qquad\qquad \cup \qquad\qquad\qquad \cup$$
$$(Z_{r-1})_{L,\sigma} \quad \cdots \quad (Z_1)_{L,\sigma} \qquad\qquad (Z_0)_{L,\sigma}$$

This property will hold automatically for all blow-up sequence functors that we construct.

34.3 (Closed embeddings). In the proof of (22) we constructed a resolution of a variety Z by choosing an embedding of Z into a smooth variety Y. In order to get a well-defined resolution, we need to know that our constructions do not depend on the embedding chosen. The key step is to ensure independence from further embeddings $Z \hookrightarrow Y \hookrightarrow X$.

We say that $\mathcal{B}$ *commutes with closed embeddings* if

$$\mathcal{B}(X, I_X, E) = j_*\,\mathcal{B}(Y, I_Y, E|_Y),$$

whenever

- $j : Y \hookrightarrow X$ is a closed embedding of smooth schemes,
- $0 \neq I_Y \subset \mathcal{O}_Y$ and $0 \neq I_X \subset \mathcal{O}_X$ are ideal sheaves such that $\mathcal{O}_X/I_X = j_*(\mathcal{O}_Y/I_Y)$, and
- E is a simple normal crossing divisor on X such that $E|_Y$ is also a simple normal crossing divisor on Y.

34.4 (Closed embeddings, weak form). Let the notation and assumptions be as in (34.3). We say that $\mathcal{B}$ *weakly commutes with closed embeddings* if

$$j^*\,\mathcal{B}(X, I_X, E) = \mathcal{B}(Y, I_Y, E|_Y).$$

The difference appears only in the proof of (35) given in (72). At the beginning of the proof we blow up various intersections of the irreducible components of E. Since these intersections are not contained in Y, this commutes with restriction to Y but it does not commute with push forward.

The strongest form of monomialization is the following.

Theorem 35 (Principalization, IV)**.** *There is a blow-up sequence functor $\mathcal{BP}$ defined on all triples (X, I, E), where X is a smooth scheme of finite type over a field of characteristic zero, $I \subset \mathcal{O}_X$ is an ideal sheaf that is not zero on any irreducible component of X and E is a simple normal crossing divisor on X. $\mathcal{BP}$ satisfies the following conditions.*

(1) *In the blow-up sequence* $\mathcal{BP}(X, I, E) =$

$$\Pi : X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X,$$
$$\cup \qquad\qquad\qquad \cup \qquad \cup$$
$$Z_{r-1} \qquad \cdots \qquad Z_1 \qquad Z_0$$

all centers of blow-ups are smooth and have simple normal crossing with E (25).

(2) *The pull-back $\Pi^* I \subset \mathcal{O}_{X_r}$ is the ideal sheaf of a simple normal crossing divisor.*

(3) $\Pi : X_r \to X$ *is an isomorphism over* $X \setminus \operatorname{cosupp} I$.

(4) $\mathcal{BP}$ *commutes with smooth morphisms (34.1) and with change of fields (34.2).*

(5) $\mathcal{BP}$ *commutes with closed embeddings (34.3) whenever $E = \emptyset$.*

Putting together the proof of (27) with (37), we obtain strong and functorial resolution.

Theorem 36 (Resolution of singularities, III). *There is a blow-up sequence functor* $\mathcal{BR}(X) =$

$$\Pi : X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X,$$
$$\cup \qquad\qquad\qquad \cup \qquad \cup$$
$$Z_{r-1} \qquad \cdots \qquad Z_1 \qquad Z_0$$

defined on all schemes X of finite type over a field of characteristic zero, satisfying the following conditions.

(1) X_r *is smooth.*

(2) $\Pi : X_r \to X$ *is an isomorphism over the smooth locus X^{ns}.*

(3) $\Pi^{-1}(\operatorname{Sing} X)$ *is a divisor with simple normal crossings.*

(4) $\mathcal{BR}$ *commutes with smooth morphisms (34.1) and with change of fields (34.2).*

Proof. First we construct $\mathcal{BR}(X)$ for affine schemes. Pick any embedding $X \hookrightarrow A$ into a smooth affine scheme such that $\dim A \geq \dim X + 2$. As in the proof of (22), the blow-up sequence for $\mathcal{BP}(A, I_X, \emptyset)$ obtained in (35) gives a blow-up sequence $\mathcal{BR}(X)$.

Before we can even consider the functoriality conditions, we need to prove that $\mathcal{BR}(X)$ is independent of the choice of the embedding $X \hookrightarrow A$.

Thus assume that $\Pi_1 : R_1(X) \to \cdots \to X$ and $\Pi_2 : R_2(X) \to \cdots \to X$ are two blow-up sequences constructed this way. Using that $\mathcal{BP}$ weakly commutes with closed embeddings (34.4), it is enough to prove uniqueness for resolutions constructed from embeddings into affine spaces $X \hookrightarrow \mathbb{A}^n$. Moreover, we are allowed to increase n anytime by taking a further embedding $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$.

As (39) shows, any two embeddings $i_1, i_2 : X \hookrightarrow \mathbb{A}^n$ become equivalent by an automorphism of $\mathbb{A}^{2n}$, which gives the required uniqueness.

Thus (34.2) for $\mathcal{BP}(A, I_X, \emptyset)$ implies (34.2) for $\mathcal{BR}(X)$ since an embedding $i : X \hookrightarrow A$ over K and $\sigma : K \hookrightarrow L$ gives another embedding $i_{\sigma,L} : X_{\sigma,L} \hookrightarrow A_{\sigma,L}$.

We can also reduce the condition (34.1) for $\mathcal{BR}(X)$ to the same condition for $\mathcal{BP}(A, I_X, \emptyset)$.

To see this, let $h : Y \to X$ be a smooth morphism, and choose any embedding $X \hookrightarrow A_X$ into a smooth affine variety. By (41), for every $y \in Y$ there is an

open neighborhood $h(y) \in A_X^0 \subset A_X$ and a smooth surjection $h_A : A_Y^0 \twoheadrightarrow A_X^0$ such that $h_A^{-1}(X \cap A_Y^0)$ is isomorphic to an open neighborhood $y \in Y^0 \subset Y$. Set $X^0 := X \cap A_X^0$. Thus, by (34.1),

$$h_A^* \, \mathcal{BP}(A_X^0, I_{X^0}, \emptyset) = \mathcal{BP}(A_Y^0, I_{Y^0}, \emptyset),$$

which shows that $h^* \, \mathcal{BR}(X^0) = \mathcal{BR}(Y^0)$. As we noted earlier, (34.1) is a local property, and thus $h^* \, \mathcal{BR}(X) = \mathcal{BR}(Y)$ as required.

We have now defined $\mathcal{BR}$ on (possibly reducible) affine schemes, and it remains to prove that one can glue together a global resolution out of these local pieces. This turns out to be a formal property of blow-up sequence functors, which we treat next. $\qquad\square$

Proposition 37. *Let $\mathcal{B}$ be a blow-up sequence functor defined on affine schemes over a field k that commutes with smooth surjections.*

Then $\mathcal{B}$ has a unique extension to a blow-up sequence functor $\overline{\mathcal{B}}$, which is defined on all schemes of finite type over k and which commutes with smooth surjections.

Proof. For any X choose an open affine cover $X = \cup U_i$, and let $X' := \coprod_i U_i$ be the disjoint union. Then X' is affine, and there is a smooth surjection $g : X' \to X$. We show that $\mathcal{B}(X')$ descends to a blow-up sequence of X.

Set $X'' := \coprod_{i \leq j} U_i \cap U_j$. (We can also think of it as the fiber product $X' \times_X X'$.) There are surjective open immersions $\tau_1, \tau_2 : X'' \to X'$, where $\tau_1|_{U_i \cap U_j} : U_i \cap U_j \to U_i$ is the first inclusion and $\tau_2|_{U_i \cap U_j} : U_i \cap U_j \to U_j$ is the second.

The blow-up sequence $\mathcal{B}(X')$ starts with blowing up $Z_0' \subset X'$, and the blow-up sequence $\mathcal{B}(X'')$ starts with blowing up $Z_0'' \subset X''$. Since $\mathcal{B}$ commutes with the τ_i, we conclude that

$$\tau_1^*(Z_0') = Z_0'' = \tau_2^*(Z_0'). \tag{37.1}$$

Since $Z_0' \subset X'$ is a disjoint union of its pieces $Z_{0i}' := Z_0' \cap U_i$, (37.1) is equivalent to saying that for every i, j

$$Z_{0i}'|_{U_i \cap U_j} = Z_{0j}'|_{U_i \cap U_j}. \tag{37.2}$$

Thus the subschemes $Z_{0i}' \subset U_i$ glue together to a subscheme $Z_0 \subset X$.

This way we obtain $X_1 := B_{Z_0} X$ such that $X_1' = X' \times_X X_1$. We can repeat the above argument to obtain the center $Z_1 \subset X_1$ and eventually get the whole blow-up sequence for X. $\qquad\square$

Warning 38. A key element of the above argument is that we need to know $\mathcal{B}$ for the disconnected affine scheme $\coprod_i U_i$.

Any resolution functor defined on connected schemes automatically extends to disconnected schemes, but for blow-up sequence functors this is not at all the case. Although the blow-ups on different connected components do not affect each other, in a resolution process we need to know in which order we perform them, see (28.1).

Besides proving resolution for nonprojective schemes and for algebraic spaces, the method of (37) is used in the proof of the principalization theorems. The inductive proof naturally produces resolution processes only locally, and this method shows that they automatically globalize.

The following lemma shows that an affine scheme has a unique embedding into affine spaces, if we stabilize the dimension.

Lemma 39. *Let X be an affine scheme and $i_1 : X \hookrightarrow \mathbb{A}^n$ and $i_2 : X \hookrightarrow \mathbb{A}^m$ two closed embeddings. Then the two embeddings into the coordinate subspaces*

$$i_1' : X \hookrightarrow \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m} \quad and \quad i_2' : X \hookrightarrow \mathbb{A}^m \hookrightarrow \mathbb{A}^{n+m}$$

are equivalent under a (nonlinear) automorphism of $\mathbb{A}^{n+m}$.

Proof. We can extend i_1 to a morphism $j_1 : \mathbb{A}^m \to \mathbb{A}^n$ and i_2 to a morphism $j_2 : \mathbb{A}^n \to \mathbb{A}^m$.

Let $\mathbf{x}$ be coordinates on $\mathbb{A}^n$ and $\mathbf{y}$ coordinates on $\mathbb{A}^m$. Then

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y} + j_2(\mathbf{x}))$$

is an automorphism of $\mathbb{A}^{n+m}$, which sends the image of i_1' to

$$\operatorname{im}\big[i_1 \times i_2 : X \to \mathbb{A}^n \times \mathbb{A}^m\big].$$

Similarly,

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} + j_1(\mathbf{y}), \mathbf{y})$$

is an automorphism of $\mathbb{A}^{n+m}$, which sends the image of i_2' to

$$\operatorname{im}\big[i_1 \times i_2 : X \to \mathbb{A}^n \times \mathbb{A}^m\big]. \quad \square$$

Aside 40. It is worthwhile to mention a local variant of (39). Let X be a scheme and $x \in X$ a point whose Zariski tangent space has dimension d. Then, for $m \geq 2d$, $x \in X$ has a unique embedding into a smooth scheme of dimension m, up to étale coordinate changes.

See [Jel87, Kal91] for affine versions.

Lemma 41. *Let $h : Y \to X$ be a smooth morphism, $y \in Y$ a point and $i : X \hookrightarrow A$ a closed embedding. Then there are open neighborhoods $y \in Y^0 \subset Y$, $f(y) \in A_X^0 \subset A_X$, $X^0 = X \cap A_X^0$; a smooth morphism $h_A : A_Y^0 \to A_X^0$; and a closed embedding $j : Y^0 \hookrightarrow A_Y^0$ such that the following diagram is a fiber product square:*

$$
\begin{array}{ccc}
Y^0 & \overset{j}{\hookrightarrow} & A_Y^0 \\
h \downarrow & \square & \downarrow h_A \\
X^0 & \overset{i}{\hookrightarrow} & A_X^0.
\end{array}
$$

Proof. We prove this over infinite fields, which is the only case that we use.

The problem is local, and thus we may assume that X, Y, A_X are affine and $Y \subset X \times \mathbb{A}^N$. If h has relative dimension d, choose a general projection $\sigma : \mathbb{A}_x^N \to \mathbb{A}_x^{d+1}$ such that $\sigma : h^{-1}(x) \to \mathbb{A}_x^{d+1}$ is finite and an embedding in a neighborhood of y. (Here we need that the residue field of x is infinite.) Thus, by shrinking Y, we may assume that Y is an open subset of a hypersurface $H \subset X \times \mathbb{A}^{d+1}$ and the first projection is smooth at $y \in H$. H is defined by an equation $\sum_I \phi_I z^I$, where the ϕ_I are regular functions on X and z denotes the coordinates on $\mathbb{A}^{d+1}$. Since $X \hookrightarrow A_X$ is a closed embedding, the ϕ_I extend to regular functions Φ_I on A_X. Set

$$A_Y := \big(\sum_I \Phi_I z^I = 0\big) \subset A_X \times \mathbb{A}^{d+1}.$$

Thus $Y \subset A_Y$ and the projection $A_Y \to A_X$ is smooth at y. Let $y \in A_Y^0 \subset A_Y$ and $A_X^0 \subset A_X$ be open sets such that the projection $h_A : A_Y^0 \to A_X^0$ is smooth and surjective. Set $Y^0 := Y \cap A_Y^0$. $\qquad\square$

The following comments on resolution for algebraic and analytic spaces are not used elsewhere in these notes.

42 (Algebraic spaces)**.** All we need to know about algebraic spaces is that étale locally they are like schemes. That is, there is a (usually nonconnected) scheme of finite type U and an étale surjection $\sigma : U \to X$. We can even assume that U is affine.

The fiber product $V := U \times_X U$ is again a scheme of finite type with two surjective, étale projection morphisms $\rho_i : V \to U$, and for all purposes one can identify the algebraic space with the diagram of schemes

$$X = \left[\rho_1, \rho_2 : V \rightrightarrows U \right]. \tag{42.1}$$

The argument of (37) applies to show that any blow-up sequence functor $\mathcal{B}$ that is defined on affine schemes over a field k and commutes with étale surjections, has a unique extension to a blow-up sequence functor $\overline{\mathcal{B}}$, which is defined on all algebraic spaces over k. (See (105) for details.) Thus we obtain the following.

Corollary 43. *The theorems (35) and (36) also hold for algebraic spaces of finite type over a field of characteristic zero.* □

44 (Analytic spaces)**.** It was always understood that a good resolution method should also work for complex, real or p-adic analytic spaces. (See [GR71] for an introduction to analytic spaces.)

The traditional methods almost all worked well locally, but globalization sometimes presented technical difficulties. We leave it to the reader to follow the proofs in this chapter and see that they all extend to analytic spaces over locally compact fields, at least locally. Once, however, we have a locally defined blow-up sequence functor that commutes with smooth surjections, the argument of (37) shows that we get a globally defined blow-up sequence functor for small neighborhoods of compact sets on all analytic spaces. Once we have a resolution functor on neighborhoods of compact sets that commutes with open embeddings, we get resolution for any analytic space that is an increasing union of its compact subsets. Thus we obtain the following.

Theorem 45. *Let K be a locally compact field of characteristic zero. There is a resolution functor $\mathcal{R} : X \to (\Pi_X : R(X) \to X)$ defined on all separable K-analytic spaces with the following properties.*

(1) *$R(X)$ is smooth.*
(2) *$\Pi : R(X) \to X$ is an isomorphism over the smooth locus X^{ns}.*
(3) *$\Pi^{-1}(\operatorname{Sing} X)$ is a divisor with simple normal crossing.*
(4) *Π_X is projective over any compact subset of X.*
(5) *$\mathcal{R}$ commutes with smooth K-morphisms.* □

Aside 46. We give an example of a normal, proper surface S over $\mathbb{C}$ that cannot be embedded into a smooth scheme.

Start with $\mathbb{P}^1 \times C$, where C is any smooth curve of genus ≥ 1. Take two points $c_1, c_2 \in C$. Blow up $(0, c_1)$ and (∞, c_2) to get $f : T \to \mathbb{P}^1 \times C$. We claim the following.

(1) The birational transforms $C_1 \subset T$ of $\{0\} \times C$ and $C_2 \subset T$ of $\{\infty\} \times C$ can be contracted, and we get a normal, proper surface $g : T \to S$.
(2) If $\mathcal{O}_C(c_1)$ and $\mathcal{O}_C(c_2)$ are independent in $\operatorname{Pic}(C)$, then S can not be embedded into a smooth scheme.

To get the first part, it is easy to check that a multiple of the birational transform of $\{1\}\times C+\mathbb{P}^1\times\{c_i\}$ on T is base point free and contracts C_i only, giving $g_i: T \to S_i$. Now $S_1 \setminus C_2$ and $S_2 \setminus C_1$ can be glued together to get $g: T \to S$.

If D is a Cartier divisor on S, then $\mathcal{O}_T(g^*D)$ is trivial on both C_1 and C_2. Therefore, $f_*(g^*D)$ is a Cartier divisor on $\mathbb{P}^1 \times C$ such that its restriction to $\{0\} \times C$ is linearly equivalent to a multiple of c_1 and its restriction to $\{\infty\} \times C$ is linearly equivalent to a multiple of c_2.

Since $\mathrm{Pic}(\mathbb{P}^1 \times C) = \mathrm{Pic}(C) \times \mathbb{Z}$ and $\mathcal{O}_C(c_1)$ and $\mathcal{O}_C(c_2)$ are independent in $\mathrm{Pic}(C)$, every Cartier divisor on S is linearly equivalent to a multiple of $\{1\} \times C$. Thus the points of $\{1\} \times C \subset S$ cannot be separated from each other by Cartier divisors on S.

Assume now that $S \hookrightarrow Y$ is an embedding into a smooth scheme. Pick a point $p \in \{1\} \times C \subset Y$, and let $p \in U \subset Y$ be an affine neighborhood. Any two points of U can be separated from each other by Cartier divisors on U. Since Y is smooth, the closure of a Cartier divisor on U is automatically Cartier on Y. Thus any two points of $U \cap S$ can be separated from each other by Cartier divisors on S, a contradiction. $\qquad\square$

An example of a toric variety with no Cartier divisors is given in [Ful93, p.65]. This again has no smooth embeddings.

4. Plan of the proof

This section contains a still somewhat informal review of the main steps of the proof. For simplicity, the role of the divisor E is ignored for now. All the definitions and theorems will be made precise later.

We need some way to measure how complicated an ideal sheaf is at a point. For the present proof a very crude measure—the order of vanishing or, simply, order—is enough.

Definition 47. Let X be a smooth variety and $0 \neq I \subset \mathcal{O}_X$ an ideal sheaf. For a point $x \in X$ with ideal sheaf m_x, we define the *order of vanishing* or *order of I* at x to be
$$\mathrm{ord}_x I := \max\{r : m_x^r \mathcal{O}_{x,X} \supset I\mathcal{O}_{x,X}\}.$$
It is easy to see that $x \mapsto \mathrm{ord}_x I$ is a constructible and upper-semi-continuous function on X.

For an irreducible subvariety $Z \subset X$, we define the *order of I* along $Z \subset X$ as
$$\mathrm{ord}_Z I := \mathrm{ord}_\eta I, \quad \text{where } \eta \in Z \text{ is the generic point.}$$
Frequently we also use the notation $\mathrm{ord}_Z I = m$ (resp., $\mathrm{ord}_Z I \geq m$) when Z is not irreducible. In this case we always assume that the order of I at every generic point of Z is m (resp., $\geq m$).

The *maximal order* of I along $Z \subset X$ is
$$\text{max-ord}_Z I := \max\{\mathrm{ord}_z I : z \in Z\}.$$
We frequently use $\text{max-ord}\, I$ to denote $\text{max-ord}_X I$.

If $I = (f)$ is a principal ideal, then the order of I at a point x is the same as the multiplicity of the hypersurface $(f = 0)$ at x. This is a simple but quite strong invariant.

In general, however, the order is a very stupid invariant. For resolution of singularities we always start with an embedding $X \hookrightarrow \mathbb{P}^N$, where N is larger than the

embedding dimension of X at any point. Thus the ideal sheaf I_X of X contains an order 1 element at every point (the local equation of a smooth hypersurface containing X), so the order of I_X is 1 at every point of X. Hence the order of I_X does not "see" the singularities of X at all. (In the proof given in Section 3.12, trivial steps reduce the principalization of the ideal sheaf of $X \subset \mathbb{P}^N$ near a point $x \in X$ to the principalization of the ideal sheaf of $X \subset P$, where $P \subset \mathbb{P}^N$ is smooth and has the smallest possible dimension locally near x. Thus we start actual work only when ord $I \geq 2$.)

There is one useful property of $\operatorname{ord}_Z I$, which is exactly what we need: the number $\operatorname{ord}_Z I$ equals the multiplicity of $\pi^* I$ along the exceptional divisor of the blow-up $\pi : B_Z X \to X$.

Definition 48 (Birational transform of ideals)**.** Let X be a smooth variety and $I \subset \mathcal{O}_X$ an ideal sheaf. For $\dim X \geq 2$ an ideal cannot be written as the product of prime ideals, but the codimension 1 primes can be separated from the rest. That is, there is a unique largest effective divisor $\operatorname{Div}(I)$ such that $I \subset \mathcal{O}_X(-\operatorname{Div}(I))$, and we can write

$$I = \mathcal{O}_X\big(-\operatorname{Div}(I)\big) \cdot I_{\mathrm{cod} \geq 2}, \quad \text{where } \operatorname{codim} \operatorname{Supp}(\mathcal{O}_X / I_{\mathrm{cod} \geq 2}) \geq 2.$$

We call $\mathcal{O}_X\big(-\operatorname{Div}(I)\big)$ the *divisorial part* of I and $I_{\mathrm{cod} \geq 2} = \mathcal{O}_X\big(\operatorname{Div}(I)\big) \cdot I$ the *codimension ≥ 2 part* of I.

Let $f : X' \to X$ be a birational morphism between smooth varieties. Assume for simplicity that I has no divisorial part, that is, $I = I_{\mathrm{cod} \geq 2}$. We are interested in the codimension ≥ 2 part of $f^* I$, called the *birational transform* of I and denoted by $f_*^{-1} I$. (It is also frequently called the weak transform in the literature.) Thus

$$f_*^{-1} I = \mathcal{O}_{X'}\big(\operatorname{Div}(f^* I)\big) \cdot f^* I.$$

We have achieved principalization iff the codimension ≥ 2 part of $f^* I$ is not there, that is, when $f_*^{-1} I = \mathcal{O}_{X'}$.

For reasons connected with (13.2), we also need another version, where we "pretend" that I has order m.

A *marked ideal sheaf* on X is a pair (I, m) where $I \subset \mathcal{O}_X$ is an ideal sheaf on X and m is a natural number.

Let $\pi : B_Z X \to X$ be the blow-up of a smooth subvariety Z and $E \subset B_Z X$ the exceptional divisor. Assume that $\operatorname{ord}_Z I \geq m$. Set

$$\pi_*^{-1}(I, m) := \big(\mathcal{O}_{B_Z X}(mE) \cdot \pi^* I, m\big),$$

and call it the *birational transform* of (I, m).

If $\operatorname{ord}_Z I = m$, then this coincides with $f_*^{-1} I$, but for $\operatorname{ord}_Z I > m$ the cosupport of $f_*^{-1}(I, m)$ also contains E. (We never use the case where $\operatorname{ord}_Z I < m$, since then $f_*^{-1}(I, m)$ is not an ideal sheaf.) One can iterate this procedure to define $f_*^{-1}(I, m)$ whenever $f : X' \to X$ is the composite of blow-ups of smooth irreducible subvarieties as above, but one has to be quite careful with this; see (63).

49 (Order reduction theorems)**.** The technical core of the proof consists of two order reduction theorems using smooth blow-ups that match the order that we work with.

Let I be an ideal sheaf with max-ord $I \leq m$. A *smooth blow-up sequence* of order m starting with (X, I) is a smooth blow-up sequence (29)

$$\Pi : (X_r, I_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, I_1) \xrightarrow{\pi_0} (X_0, I_0) = (X, I),$$

where each $\pi_i : X_{i+1} \to X_i$ is a smooth blow-up with center $Z_i \subset X_i$, the I_i are defined recursively by the formula $I_{i+1} := (\pi_i)_*^{-1} I_i$ and $\mathrm{ord}_{Z_i} I_i = m$ for every $i < r$.

A *blow-up sequence* of order $\geq m$ starting with a marked ideal (X, I, m) is defined analogously, except we use the recursion formula $(I_{i+1}, m) := (\pi_i)_*^{-1}(I_i, m)$ and we require $\mathrm{ord}_{Z_i} I_i \geq m$ for every $i < r$.

Using these notions, the inductive versions of the main results are the following.

49.1 (Order reduction for ideals). *Let X be a smooth variety, $0 \neq I \subset \mathcal{O}_X$ an ideal sheaf and $m = \mathrm{max\text{-}ord}\, I$. By a suitable blow-up sequence of order m we eventually get $f : X' \to X$ such that $\mathrm{max\text{-}ord}\, f_*^{-1} I < m$.*

49.2 (Order reduction for marked ideals). *Let X be a smooth variety, $0 \neq I \subset \mathcal{O}_X$ an ideal sheaf and $m \leq \mathrm{max\text{-}ord}\, I$ a natural number. By a suitable blow-up sequence of order $\geq m$, we eventually get $f : X' \to X$ such that $\mathrm{max\text{-}ord}\, f_*^{-1}(I, m) < m$.*

We prove these theorems together in a spiraling induction with two main reduction steps.

<div style="border:1px solid">

order reduction for marked ideals in dimension $n - 1$

$\Downarrow$

order reduction for ideals in dimension n

$\Downarrow$

order reduction for marked ideals in dimension n

</div>
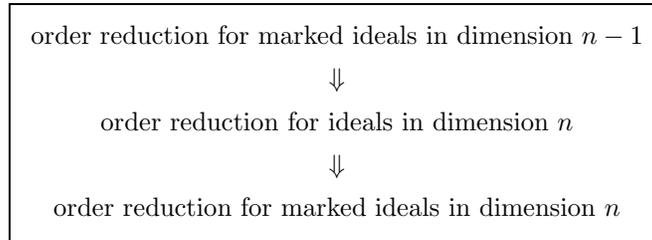
The two steps are independent and use different methods.

The second implication is relatively easy and has been well understood for a long time. We leave it to Section 3.13.

Here we focus on the proof of the harder part, which is the first implication.

50 (The heart of the proof). Methods to deal with Problems (13.3–5) form the key steps of the proof. My approach is to break apart the traditional inductive proof. The problems can be solved independently but only for certain ideals. Then we need one more step to show that order reduction for an arbitrary ideal is equivalent to order reduction for an ideal with all the required good properties.

50.1 (Maximal contact). This deals with (13.3) by showing that for suitable hypersurfaces $H \subset X$ every step of an order reduction algorithm for (X, I) with $m = \mathrm{max\text{-}ord}\, I$ is also a step of an order reduction algorithm for $(H, I|_H, m)$. This is explained in (51) and completed in Section 3.8.

50.2 (*D*-balanced ideals). Problem (13.4) has a solution for certain ideals only. For the so-called *D*-balanced ideals, the converse of maximal contact theory holds. That is, for every hypersurface $S \subset X$, every order reduction step for $(S, I|_S, m)$ is also an order reduction step for (X, I). This is outlined in (52) with all details in Section 3.9.

50.3 (MC-invariant ideals). The solution of (13.5) requires the consideration of MC-invariant ideals. For these, all hypersurfaces of maximal contact are locally analytically isomorphic, with an isomorphism preserving the ideal I. See (53), with full proofs in Section 3.10.

50.4 (Tuning of ideals). It remains to show that order reduction for an arbitrary ideal I is equivalent to order reduction for an ideal $W(I)$, which is both D-balanced and MC-invariant. This turns out to be surprisingly easy; see (54) and Section 3.11.

50.5 (Final assembly). The main remaining problem is that a hypersurface of maximal contact can be found only locally, not globally. The local pieces are united in Section 3.12, where we also take care of the divisor E, which we have ignored so far.

Let us now see these steps in more detail.

51 (Maximal contact). Following the examples (11) and (12), given X and I with $m = \text{max-ord}\,I$, we would like to find a smooth hypersurface $H \subset X$ such that order reduction for I follows from order reduction for $(I|_H, m)$.

As we noted in (13.3), first, we have to ensure that the points where the birational transform of I has order $\geq m$ stay on the birational transform of H all the time. That is, we want to achieve the following.

51.1 (Going-down property of maximal contact). Restriction (30.2.) from X to H gives an injection

$$\boxed{\begin{array}{c} \text{blow-up sequences of order } m \text{ for } (X,I) \\ \bigcap \\ \text{blow-up sequences of order} \geq m \text{ for } (H, I|_H, m) \end{array}}$$

If this holds, then we say that H is a hypersurface of *maximal contact*. At least locally these are easy to find using derivative ideals.

Derivations of a smooth variety X form a sheaf Der_X, locally generated by the usual partials $\partial/\partial x_i$. For an ideal sheaf I, let $D(I)$ denote the ideal sheaf generated by all derivatives of local sections of I. We can define higher-derivative ideals inductively by the rule $D^{i+1}(I) := D\big(D^i(I)\big)$.

If $m = \text{max-ord}\,I$, we are especially interested in the largest nontrivial derivative ideal. It is also called the *ideal of maximal contacts*

$$MC(I) := D^{m-1}(I) = \left(\frac{\partial^{m-1} f}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}} : f \in I,\ \sum c_i = m - 1 \right).$$

51.2 (Local construction of maximal contact). *For a point $p \in X$ with $m = \text{ord}_p\,I$, let $h \in MC(I)$ be any local section with $\text{ord}_p\,h = 1$. Then $H := (h = 0)$ is a hypersurface of maximal contact in an open neighborhood of p.*

In general, hypersurfaces of maximal contact do not exist globally, and they are not unique locally. We deal with these problems later.

52 (*D*-balanced ideals). It is harder to deal with (13.4). No matter how we choose the hypersurface of maximal contact H, sometimes the restriction $(I|_H, m)$ is "more singular" than I, in the sense that order reduction for $(I|_H, m)$ may involve blow-ups that are not needed for any order reduction procedure of I; see (82).

There are, however, some ideals for which this problem does not happen. To define these, we again need to consider derivatives.

If $\mathrm{ord}_p\, f = m$, then typically $\mathrm{ord}_p(\partial f/\partial x_i) = m - 1$, so a nontrivial ideal is never D-closed. The best one can hope for is that I is D-closed, after we "correct for the lowering of the order."

An ideal I with $m = \mathrm{max\text{-}ord}\, I$ is called D-*balanced* if

$$\left(D^i(I)\right)^m \subset I^{m-i} \quad \forall\, i < m.$$

Such ideals behave very well with respect to restriction to smooth subvarieties and smooth blow ups.

52.1 (Going-up property of D-balanced ideals). Let I be a D-balanced ideal with $m = \mathrm{max\text{-}ord}\, I$. Then for any smooth hypersurface $S \subset X$ such that $S \not\subset \mathrm{cosupp}\, I$, push-forward (30.3.) from S to X gives an injection

$$\boxed{\begin{array}{c} \text{blow-up sequences of order } m \text{ for } (X, I) \\[4pt] \bigcup \\[4pt] \text{blow-up sequences of order} \geq m \text{ for } (S, I|_S, m) \end{array}}$$

Example 52.2. Start with the double point ideal $I = (xy - z^n)$. Restricting to $S = (x = 0)$ creates an n-fold line, and blowing up this line is not an order 2 blow-up for I.

We can see that

$$I + D(I)^2 = (xy, x^2, y^2, xz^{n-1}, yz^{n-1}, z^n)$$

is D-balanced. If we restrict $I + D(I)^2$ to $(x = 0)$, we get the ideal (y^2, yz^{n-1}, z^n). It is easy to check that the whole resolution of S is correctly predicted by order reduction for (y^2, yz^{n-1}, z^n).

Putting (51.1) and (52.1) together, we get the first dimension reduction result.

Corollary 52.3. (Maximal contact for D-balanced ideals). *Let I be a D-balanced ideal with $m = \mathrm{max\text{-}ord}\, I$ and $H \subset X$ a smooth hypersurface of maximal contact. Then we have an equivalence*

$$\boxed{\begin{array}{c} \textit{blow-up sequences of order } m \textit{ for } (X, I) \\[4pt] \| \\[4pt] \textit{blow-up sequences of order} \geq m \textit{ for } (H, I|_H, m) \end{array}}$$

This equivalence suggests that the choice of H should not be important at all. However, in order to ensure functoriality we have to choose a particular resolution. Thus we still need to show that our particular choices are independent of H. A truly "canonical" resolution process would probably take care of such problems automatically, but it seems that one has to make at least some artificial choices.

53 (MC-invariant ideals). Dealing with (13.5) is again possible only for certain ideals.

We say that I is *maximal contact invariant* or *MC-invariant* if

$$MC(I) \cdot D(I) \subset I. \tag{53.1}$$

Written in the equivalent form

$$D^{m-1}(I) \cdot D(I) \subset I, \tag{53.2}$$

it is quite close in spirit to the D-balanced condition. The expected order of $D^{m-1}(I) \cdot D(I)$ is m, so it is sensible to require inclusion. There is no need to correct for the change of order first.

For MC-invariant ideals the hypersurfaces of maximal contact are still not unique, but different choices are equivalent under local analytic isomorphisms (55).

53.3 (Formal uniqueness of maximal contact). *Let I be an MC-invariant ideal sheaf on X and $H_1, H_2 \subset X$ two hypersurfaces of maximal contact through a point $x \in X$. Then there is a local analytic automorphism (55) $\phi : (x \in \hat{X}) \to (x \in \hat{X})$ such that*

(i) *$\phi^{-1}(\hat{H}_1) = \hat{H}_2$,*
(ii) *$\phi^* \hat{I} = \hat{I}$, and*
(iii) *ϕ is the identity on* cosupp $\hat{I}$.

54 (Tuning of ideals)**.** Order reduction using dimension induction is now in quite good shape for ideals that are both D-balanced and MC-invariant.

The rest is taken care of by "tuning" the ideal I first. (I do not plan to give a precise meaning to the word "tuning." The terminology follows [Wło05]. The notion of tuning used in [EH02] is quite different.) There are many ways to tune an ideal; here is one of the simplest ones.

To an ideal I of order m, we would like to associate the ideal generated by all products of derivatives of order at least m. The problem with this is that if f has order m, then $\partial f / \partial x_i$ has order $m - 1$, and so we are able to add $(\partial f / \partial x_i)^2$ (which has order $2m - 2$), but we really would like to add $(\partial f / \partial x_i)^{m/(m-1)}$ (which should have order m in any reasonable definition).

We can avoid these fractional exponent problems by working with all products of derivatives whose order is sufficiently divisible. For instance, the condition (order) $\geq m!$ works.

Enriching an ideal with its derivatives was used by Hironaka [Hir77] and then developed by Villamayor [Vil89]. A larger ideal is introduced in [Wło05]. The ideal $W(I)$ introduced below is even larger, and this largest choice seems more natural to me. That is, we set

$$W(I) := \left(\prod_{j=0}^{m} \left(D^j(I)\right)^{c_j} : \sum (m-j)c_j \geq m! \right) \subset \mathcal{O}_X. \qquad (54.1)$$

The ideal $W(I)$ has all the properties that we need.

Theorem 54.2. (Well-tuned ideals). Let X be a smooth variety, $0 \neq I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Then

(i) $\text{max-ord}\, W(I) = m!$,
(ii) $W(I)$ is D-balanced,
(iii) $W(I)$ is MC-invariant, and
(iv) there is an equivalence

> blow-up sequences of order m for (X, I)
>
> $||$
>
> blow-up sequences of order $m!$ for $(X, W(I))$

54.3. It should be emphasized that there are many different ways to choose an ideal with the properties of $W(I)$ as above, but all known choices have rather high order.

I chose the order $m!$ for notational simplicity, but one could work with any multiple of $\operatorname{lcm}(1, 2, \ldots, m)$ instead. The smallest choice would be $\operatorname{lcm}(1, 2, \ldots, m)$, which is roughly like e^m. As discussed in (7.2), this is still too big for effective computations. Even if we fix the order to be $m!$, many choices remain.

54.4. Similar constructions are also considered by Kawanoue [Kaw06] and by Villamayor [Vil06].

Definition 55 (Completions). This is the only piece of commutative algebra that we use.

For a local ring (R, m) its *completion* in the m-adic topology is denoted by $\hat{R}$; cf. [AM69, Chap.10]. If X is a k-variety and $x \in X$, then we denote by $\hat{X}_x$ or by $\hat{X}$ the completion of X at x, which is $\operatorname{Spec}_k \hat{\mathcal{O}}_{x,X}$.

We say that $x \in X$ and $y \in Y$ are *formally* isomorphic if $\hat{X}_x$ is isomorphic to $\hat{Y}_y$.

We need Krull's intersection theorem (cf. [AM69, 10.17]), which says that for an ideal I in a Noetherian local ring (R, m) we have

$$I = \cap_{s=1}^{\infty} (I + m^s).$$

In geometric language this implies that if $Z, W \subset X$ are two subschemes such that $\hat{Z}_x = \hat{W}_x$, then there is an open neighborhood $x \in U \subset X$ such that $Z \cap U = W \cap U$.

If $p \in X$ is closed, then $\hat{\mathcal{O}}_{p,X} \cong k(p)[[x_1, \ldots, x_n]]$, where $x_1, \ldots, x_n$ are local coordinates. If $k(p) = k$ or, more generally, when there is a *field of representatives* (that is, a subfield $k' \subset \hat{\mathcal{O}}_{p,X}$ isomorphic to $k(p)$), this is proved in [Sha94, II.2]. In characteristic zero one can find k' as follows. The finite field extension $k(p)/k$ is generated by a simple root of a polynomial $f(y) \in k[y] \subset \hat{\mathcal{O}}_{p,X}[y]$. Modulo the maximal ideal, $f(y)$ has a linear factor by assumption, and thus by the general Hensel lemma $f(y)$ has a linear factor and hence a root $\alpha \in \hat{\mathcal{O}}_{p,X}$. Then $k' = k(\alpha)$ is the required subfield. (Note that usually one cannot find such $k' \subset \mathcal{O}_{p,X}$, and so the completion is necessary.)

Remark 56. By the approximation theorem of [Art69], $x \in X$ and $y \in Y$ are formally isomorphic iff there is a $z \in Z$ and étale morphisms

$$(x \in X) \leftarrow (z \in Z) \rightarrow (y \in Y).$$

This implies that any resolution functor that commutes with étale morphisms also commutes with formal isomorphisms.

Our methods give resolution functors that commute with formal isomorphisms by construction, so we do not need to rely on [Art69].

Aside 57 (Maximal contact in positive characteristic). Maximal contact, in the form presented above, works in positive characteristic as long as the order of the ideal is less than the characteristic but fails in general. In some cases there is no smooth hypersurface at all that contains the set of points where the order is maximal. The following example is taken from [Nar83]. In characteristic 2 consider

$$X := (x^2 + yz^3 + zw^3 + y^7 w = 0) \subset \mathbb{A}^4.$$

The maximal multiplicity is 2, and the singular locus is given by

$$x^2 + yz^3 + zw^3 + y^7w = z^3 + y^6w = yz^2 + w^3 = zw^2 + y^7 = 0.$$

It contains the monomial curve

$$C := \mathrm{im}[t \mapsto (t^{32}, t^7, t^{19}, t^{15})]$$

(in fact, it is equal to it). C is not contained in any smooth hypersurface. Indeed, assume that $(F = 0)$ is a hypersurface containing C that is smooth at the origin. Then one of x, y, z, w appears linearly in F and $F(t^{32}, t^7, t^{19}, t^{15}) \equiv 0$. The linear term gives a nonzero t^m for some $m \in \{32, 7, 19, 15\}$, which must be canceled by another term t^m. Thus we can write $m = 32a + 7b + 19c + 15d$, where $a + b + c + d \geq 2$ and $a, b, c, d \geq 0$. This is, however, impossible since none of the numbers $32, 7, 19, 15$ is a positive linear combination of the other three.

5. BIRATIONAL TRANSFORMS AND MARKED IDEALS

58 (Birational transform of ideals). Let X be a smooth scheme over a field k, $Z \subset X$ a smooth subscheme and $\pi : B_Z X \to X$ the blow-up with exceptional divisor $F \subset B_Z X$. Let $Z = \cup Z_j$ and $F = \cup F_j$ be the irreducible components.

Let $I \subset \mathcal{O}_X$ be an ideal sheaf, and set $\mathrm{ord}_{Z_j} I = m_j$. Then $\pi^* I \subset \mathcal{O}_{B_Z X}$ vanishes along F_j with multiplicity m_j, and we aim to remove the ideal sheaf $\mathcal{O}_{B_Z X}(-\sum m_j F_j)$ from $\pi^* I$. That is, define the *birational transform* (also called the controlled transform or weak transform in the literature) of I by the formula

$$\pi_*^{-1} I := \mathcal{O}_{B_Z X}(\textstyle\sum_j m_j F_j) \cdot \pi^* I \subset \mathcal{O}_{B_Z X}. \tag{58.1}$$

This is consistent with the definition given in (49) for the case $I = I_{\mathrm{cod} \geq 2}$.

Warning. If $Z \subset X$ is a smooth divisor, then the blow-up is trivial. Hence $\pi_{Z,X} : B_Z X \cong X$ is the identity map, and

$$\pi_*^{-1} I := \mathcal{O}_X(\textstyle\sum_j m_j Z_j) \cdot I$$

depends not only on $\pi = \pi_{Z,X}$ but also on the center Z of the blow-up. Unfortunately, I did not find any good way to fix this notational inconsistency.

One problem we have to deal with in resolutions is that if $Z \subset H \subset X$ is a smooth hypersurface with birational transform $B_Z H \subset B_Z X$ and projection $\pi_H : B_Z H \to H$, then restriction to H does not commute with taking birational transform. That is,

$$(\pi_H)_*^{-1}(I|_H) \supset (\pi_*^{-1} I)|_{B_Z H}, \tag{58.2}$$

but equality holds only if $\mathrm{ord}_Z I = \mathrm{ord}_Z(I|_H)$.

The next definition is designed to remedy this problem. We replace the ideal sheaf I by a pair (I, m), where m keeps track of the order of vanishing that we pretend to have. The advantage is that we can redefine the notion of birational transform to achieve equality in (58.2).

Definition 59. Let X be a smooth scheme. A *marked function* on X is a pair (f, m), where f is a regular function on (some open set of) X and m a natural number.

A *marked ideal sheaf* on X is a pair (I, m), where $I \subset \mathcal{O}_X$ is an ideal sheaf on X and m a natural number.

The *cosupport* of (I, m) is defined by

$$\operatorname{cosupp}(I, m) := \{x \in X : \operatorname{ord}_x I \geq m\}.$$

The *product* of marked functions or marked ideal sheaves is defined by

$$(f_1, m_1) \cdot (f_2, m_2) := (f_1 f_2, m_1 + m_2), \quad (I_1, m_1) \cdot (I_2, m_2) := (I_1 I_2, m_1 + m_2).$$

The *sum* of marked functions or marked ideal sheaves is only sensible when the markings are the same:

$$(f_1, m) + (f_2, m) := (f_1 + f_2, m) \quad \text{and} \quad (I_1, m) + (I_2, m) := (I_1 + I_2, m).$$

The cosupport has the following elementary properties:

(1) if $I \subset J$ then $\operatorname{cosupp}(I, m) \supset \operatorname{cosupp}(J, m)$,
(2) $\operatorname{cosupp}(I_1 I_2, m_1 + m_2) \supset \operatorname{cosupp}(I_1, m_1) \cap \operatorname{cosupp}(I_2, m_2)$,
(3) $\operatorname{cosupp}(I, m) = \operatorname{cosupp}(I^c, mc)$,
(4) $\operatorname{cosupp}(I_1 + I_2, m) = \operatorname{cosupp}(I_1, m) \cap \operatorname{cosupp}(I_2, m)$.

Definition 60. Let X be a smooth variety, $Z \subset X$ a smooth subvariety and $\pi : B_Z X \to X$ the blow-up with exceptional divisor $F \subset B_Z X$. Let (I, m) be a marked ideal sheaf on X such that $m \leq \operatorname{ord}_Z I$. In analogy with (58) we define the *birational transform* of (I, m) by the formula

$$\pi_*^{-1}(I, m) := \big(\mathcal{O}_{B_Z X}(mF) \cdot \pi^* I, m\big). \tag{60.1}$$

Informally speaking, we use the definition (58.2), but we "pretend that $\operatorname{ord}_Z I = m$."

As in (58), it is worth calling special attention to the case where Z has codimension 1 in X. Then $B_Z X \cong X$, and so scheme-theoretically there is no change. However, the vanishing order of $\pi_*^{-1}(I, m)$ along Z is m less than the vanishing order of I along Z.

In order to do computations, choose local coordinates $(x_1, \ldots, x_n)$ such that $Z = (x_1 = \cdots = x_r = 0)$. Then

$$y_1 = \tfrac{x_1}{x_r}, \ldots, y_{r-1} = \tfrac{x_{r-1}}{x_r}, \ y_r = x_r, \ldots, y_n = x_n \tag{60.2}$$

give local coordinates on a chart of $B_Z X$, and we define

$$\pi_*^{-1}(f, m) := \big(y_r^{-m} f(y_1 y_r, \ldots, y_{r-1} y_r, y_r, \ldots, y_n), m\big). \tag{60.3}$$

This formula is the one we use to compute with blow-ups, but it is coordinate system dependent. As we change coordinates, the result of π_*^{-1} changes by a unit. So we are free to use π_*^{-1} to compute the birational transform of ideal sheaves, but one should not use it for individual functions, whose birational transform cannot be defined (as a function).

The following lemmas are easy.

Lemma 61. *Let X be a smooth variety, $Z \subset X$ a smooth subvariety, $\pi : B_Z X \to X$ the blow-up and $I \subset \mathcal{O}_X$ an ideal sheaf. Assume that $\operatorname{ord}_Z I = \operatorname{max-ord} I$. Then*

$$\operatorname{max-ord} \pi_*^{-1} I \leq \operatorname{max-ord} I.$$

Proof. Choose local coordinates as above, and pick $f(x_1, \ldots, x_n) \in I$ such that $\operatorname{ord}_p f = \operatorname{max-ord} I = m$. Its birational transform is computed as

$$\pi_*^{-1} f = y_r^{-m} f(y_1 y_r, \ldots, y_{r-1} y_r, y_r, \ldots, y_n).$$

Since $f(x_1, \ldots, x_n)$ contains a monomial of degree m, the corresponding monomial in $f(y_1 y_r, \ldots, y_{r-1} y_r, y_r, \ldots, y_n)$ has degree $\leq 2m$, and thus in $\pi_*^{-1} f$ we get a monomial of degree $\leq 2m - m = m$.

This shows that $\mathrm{ord}_{p'} \pi_*^{-1} I \leq m$, where $p' \in B_Z X$ denotes the origin of the chart we consider. Performing a linear change of the $(x_1, \ldots, x_r)$-coordinates moves the origin of the chart, and every preimage of p appears as the origin after a suitable linear change. Thus our computation applies to all points of the exceptional divisor of $B_Z X$. $\qquad\square$

Lemma 62. *Let the notation be as in (61). Let $Z \subsetneq H \subset X$ be a smooth hyper-surface with birational transform $B_Z H \subset B_Z X$ and projection $\pi_H : B_Z H \to H$. If $m \leq \mathrm{ord}_Z I$ and $I|_H \neq 0$, then*

$$(\pi_H)_*^{-1}(I|_H, m) = \left(\pi_*^{-1}(I, m)\right)|_{B_Z H}.$$

Proof. Again choose coordinates and assume that $H = (x_1 = 0)$. Working with the chart as in (60.2), the birational transform of H is $(y_1 = 0)$, and we see that it does not matter whether we set first $x_1 = 0$ and compute the transform or first compute the transform and then set $y_1 = 0$. We still need to contemplate what happens in the chart

$$z_1 = x_1, z_2 = \tfrac{x_2}{x_1}, \ldots, \ z_r = \tfrac{x_r}{x_1}, \ z_{r+1} = x_{r+1}, \ldots, z_n = x_n.$$

This chart, however, does not contain any point of the birational transform of H, so it does not matter. $\qquad\square$

Note that (62) fails if $Z = H$. In this case $I|_H$ is the zero ideal, π_Z is an isomorphism, and we have only the bad chart, which we did not need to consider in the proof above. Because of this, we will have to consider codimension 1 subsets of $\mathrm{cosupp}\, I$ separately.

Warning 63. Note that, while the birational transform of an ideal with $I = I_{\mathrm{cod} \geq 2}$ is defined for an arbitrary birational morphism (58), we have defined the birational transform of a marked ideal only for a single smooth blow-up (60). This can be extended to a sequence of smooth blow-ups, but one has to be very careful. Let

$$\Pi : X' = X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X \qquad (63.1)$$

be a smooth blow-up sequence. We can inductively define the birational transforms of the marked ideal (I, m) by

(1) $(I_0, m) := (I, m)$, and
(2) $(I_{j+1}, m) := (\pi_j)_*^{-1}(I_j, m)$ as in (60).

At the end we get (I_r, m), which I rather sloppily also denote by $\Pi_*^{-1}(I, m)$.

It is very important to keep in mind that this notation assumes that we have a particular blow-up sequence in mind. That is, $\Pi_*^{-1}(I, m)$ depends not only on the morphism Π but on the actual sequence of blow-ups we use to get it.

Consider, for instance, the blow-ups

$$
\begin{array}{ccccc}
\Pi : X_2 & \xrightarrow{\pi_1} & X_1 = B_p X_0 & \xrightarrow{\pi_0} & X_0, \\
\| & & & & \| \\
\Sigma : X_2' & \xrightarrow{\sigma_1} & X_1' = B_C X_0 & \xrightarrow{\sigma_0} & X_0
\end{array}
$$

introduced in (33).

Let us compute the birational transforms of $(I, 1)$, where $I := I_C$. The first blow-up sequence gives

$$
\begin{aligned}
(\pi_0)_*^{-1}(I, 1) &= \mathcal{O}_{X_1}(E_0) \cdot \pi_0^* I \quad \text{and} \\
(\pi_1)_*^{-1}\big((\pi_0)_*^{-1}(I, 1)\big) &= \mathcal{O}_{X_2}(E_1) \cdot \pi_1^*\big((\pi_0)_*^{-1}(I, 1)\big) \\
&= \mathcal{O}_{X_2}(E_0 + E_1) \cdot \Pi^* I.
\end{aligned}
$$

On the other hand, the second blow-up sequence gives

$$
\begin{aligned}
(\sigma_0)_*^{-1}(I, 1) &= \mathcal{O}_{X_1'}(E_0') \cdot \sigma_0^* I \quad \text{and} \\
(\sigma_1)_*^{-1}\big((\sigma_0)_*^{-1}(I, 1)\big) &= \mathcal{O}_{X_2'}(E_1') \cdot \sigma_1^*\big((\sigma_0)_*^{-1}(I, 1)\big) \\
&= \mathcal{O}_{X_2'}(E_0' + 2E_1') \cdot \Sigma^* I
\end{aligned}
$$

since $\sigma_1^* \mathcal{O}_{X_1'}(E_0') = \mathcal{O}_{X_2'}(E_0' + E_1')$.

Thus $\Pi_*^{-1}(I, 1) \neq \Sigma_*^{-1}(I, 1)$, although $\Pi = \Sigma$.

6. The inductive setup of the proof

In this section we set up the final notation and state the main order reduction theorems.

Notation 64. For the rest of the chapter, (X, I, E) or (X, I, m, E) denotes a *triple*[1], where

(1) X is a smooth, equidimensional (possibly reducible) scheme of finite type over a field of characteristic zero,
(2) $I \subset \mathcal{O}_X$ (resp., (I, m)) is a coherent ideal sheaf (resp., coherent marked ideal sheaf), which is nonzero on every irreducible component of X, and
(3) $E = (E^1, \ldots, E^s)$ is an ordered set of smooth divisors on X such that $\sum E^i$ is a simple normal crossing divisor. Each E^i is allowed to be reducible or empty.

The divisor E plays an ancillary role as a device that keeps track of the exceptional divisors that we created and of the order in which we created them. As we saw in (6.3), one has to carry along some information about the resolution process.

As we observed in (6.2) and (28.1), it is necessary to blow up disjoint subvarieties simultaneously. Thus we usually do get reducible smooth divisors E^j.

Definition 65. Given (X, I, E) with max-ord $I = m$, a *smooth blow-up* of order m is a smooth blow-up $\pi : B_Z X \to X$ with *center* Z such that

(1) $Z \subset X$ has simple normal crossings only with E, and
(2) $\mathrm{ord}_Z I = m$.

The *birational transform* of (X, I, E) under the above blow-up is

$$
\pi_*^{-1}(X, I, E) = \big(B_Z X, \pi_*^{-1} I, \pi_{\mathrm{tot}}^{-1}(E)\big).
$$

Here $\pi_*^{-1} I$ is the birational transform of I as defined in (58), and $\pi_{\mathrm{tot}}^{-1}(E)$ consists of the birational transform of E (with the same ordering as before) plus the exceptional divisor $F \subset B_Z X$ added as the last divisor. It is called the *total transform* of E. (If π is a trivial blow-up, then $\pi_{\mathrm{tot}}^{-1}(E) = E + Z$.)

A *smooth blow-up* of (X, I, m, E) is a smooth blow-up $\pi : B_Z X \to X$ such that

(1') $Z \subset X$ has simple normal crossings only with E, and
(2') $\mathrm{ord}_Z I \geq m$.

---

[1] I consider the pair (I, m) as one item, so (X, I, m, E) is still a triple.

The *birational transform* of (X, I, m, E) under the above blow-up is defined as

$$\pi_*^{-1}(X, I, m, E) = \big(B_Z X, \pi_*^{-1}(I, m), \pi_{\text{tot}}^{-1}(E)\big).$$

Definition 66. A *smooth blow-up sequence* of order m and of length r starting with (X, I, E) such that max-ord $I = m$ is a smooth blow-up sequence (30)

$$\Pi : (X_r, I_r, E_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),$$

where

(1) the (X_i, I_i, E_i) are defined recursively by the formula

$$(X_{i+1}, I_{i+1}, E_{i+1}) := \big(B_{Z_i} X_i, (\pi_i)_*^{-1} I_i, (\pi_i)_{\text{tot}}^{-1} E_i\big),$$

(2) each $\pi_i : X_{i+1} \to X_i$ is a smooth blow-up with center $Z_i \subset X_i$ and exceptional divisor $F_{i+1} \subset X_{i+1}$,
(3) for every i, $Z_i \subset X_i$ has simple normal crossings with E_i, and
(4) for every i, $\text{ord}_{Z_i} I_i = m$.

Similarly, a *smooth blow-up sequence* of order $\geq m$ and of length r starting with (X, I, m, E) is a smooth blow-up sequence

$$\Pi : (X_r, I_r, m, E_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, m, E_1) \xrightarrow{\pi_0} (X_0, I_0, m, E_0) = (X, I, m, E),$$

where

(1′) the (X_i, I_i, m, E_i) are defined recursively by the formula

$$(X_{i+1}, I_{i+1}, m, E_{i+1}) := \big(B_{Z_i} X_i, (\pi_i)_*^{-1} (I_i, m), (\pi_i)_{\text{tot}}^{-1} E_i\big),$$

(2′–3′) the sequence satisfies (2) and (3) above, and
(4′) for every i, $\text{ord}_{Z_i} I_i \geq m$.

As we noted in (60), we allow the case where $Z_i \subset X_i$ has codimension 1. In this case π_{i+1} is an isomorphism, but $I_{i+1} \neq I_i$.

We also use the notation

$$\begin{aligned} \Pi_*^{-1}(X, I, E) & := & \big(X_r, \Pi_*^{-1} I, \Pi_{\text{tot}}^{-1}(E)\big) \\ & := & (X_r, I_r, E_r), \end{aligned}$$

but keep in mind that, as we saw in (63), this depends on the whole blow-up sequence and not only on Π.

We also enrich the definition of *blow-up sequence functors* considered in (31). From now on, we consider functors $\mathcal{B}$ such that $\mathcal{B}(X, I, E)$ (resp., $\mathcal{B}(X, I, m, E)$) is a blow-up sequence starting with (X, I, E) (resp., (X, I, m, E)) as above. That is, from now on we consider the sheaves I_i and the divisors E_i as part of the functor. Since these are uniquely determined by (X, I, E) and the blow-ups π_i, this is a minor notational change.

Remark 67. The difference between the marked and unmarked versions is significant, since the birational transforms of the ideals are computed differently.

There is one case, however, when one can freely pass between the two versions. If I is an ideal with max-ord $I = m$, then in any blow-up sequence of order $\geq m$ starting with (X, I, m, E), max-ord $I_i \leq m$ by (61), and so every blow-up has order $= m$. Thus, by deleting m, we automatically get a blow-up sequence of order m starting with (X, I, E). The converse also holds.

We can now state the two main technical theorems that combine to give an inductive proof of resolution.

Theorem 68 (Order reduction for ideals). *For every m there is a smooth blow-up sequence functor $\mathcal{BO}_m$ of order m (31) that is defined on triples (X, I, E) with max-ord $I \leq m$ such that if $\mathcal{BO}_m(X, I, E) =$*

$$\Pi : (X_r, I_r, E_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),$$

then

  (1) max-ord $I_r < m$, and
  (2) $\mathcal{BO}_m$ commutes with smooth morphisms (34.1) and with change of fields (34.2).

In our examples, the case max-ord $I < m$ is trivial, that is, $X_r = X$.

Theorem 69 (Order reduction for marked ideals). *For every m there is a smooth blow-up sequence functor $\mathcal{BMO}_m$ of order $\geq m$ (31) that is defined on triples (X, I, m, E) such that if $\mathcal{BMO}_m(X, I, m, E) =$*

$$\Pi : (X_r, I_r, m, E_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, m, E_1) \xrightarrow{\pi_0} (X_0, I_0, m, E_0),$$

then

  (1) max-ord $I_r < m$, and
  (2) $\mathcal{BMO}_m$ commutes with smooth morphisms (34.1) and also with change of fields (34.2).

70 (Main inductive steps of the proof). We prove (68) and (69) together in two main reduction steps.

$$
\boxed{
\begin{array}{c}
\text{(69) in dimensions } \leq n - 1 \\
\Downarrow \qquad\qquad \text{(70.1)} \\
\text{(68) in dimension } n \\
\Downarrow \qquad\qquad \text{(70.2)} \\
\text{(69) in dimension } n
\end{array}
}
$$

The easier part is (70.2). Its proof is given in Section 3.13. Everything before that is devoted to proving (70.1).

We can start the induction with the case $\dim X = 0$. Here $I = \mathcal{O}_X$ since I is assumed nonzero on every irreducible component of X. Everything is resolved without blow-ups.

The case $\dim X = 1$ is also uninteresting. The cosupport of an ideal sheaf is a Cartier divisor and our algorithm tells us to blow up $Z := \mathrm{cosupp}(I, m)$. In the unmarked case $m = $ max-ord I. After one blow-up I is replaced by $I' := I \otimes \mathcal{O}_X(Z)$ which has order $< m$. In the marked case max-ord$\left(I \otimes \mathcal{O}_X(Z)\right) < $ max-ord I. Thus, after finitely many steps, the maximal order drops below m.

The 2-dimensional case is quite a bit more involved since it includes the resolution of plane curve singularities (essentially as in Section 1.10) and the principalization of ideal sheaves studied in Section 1.9.

71 (From principalization to resolution). As we saw in the proof of (27), one can prove the existence of resolutions for quasi-projective schemes using (68), but it is not clear that the resolution is independent of the projective embedding chosen. In order to prove it, we establish two additional properties of the functors $\mathcal{BO}$ and $\mathcal{BMO}$.

Claim 71.1. $\mathcal{BMO}_m(X, I, m, \emptyset) = \mathcal{BO}_m(X, I, \emptyset)$ if $m = \max\text{-ord}\, I$.

Claim 71.2. Let $\tau : Y \hookrightarrow X$ be a closed embedding of smooth schemes and $J \subset \mathcal{O}_Y$ and $I \subset \mathcal{O}_X$ ideal sheaves such that J is nonzero on every irreducible component of Y and $\tau_*(\mathcal{O}_Y/J) = \mathcal{O}_X/I$. Then

$$\mathcal{BMO}_1(X, I, 1, \emptyset) = \tau_* \mathcal{BMO}_1(Y, J, 1, \emptyset).$$

In both of these claims we assume that $E = \emptyset$. One can easily extend (71.1) to arbitrary E, by slightly changing the definition (110). The situation with (71.2) is more problematic. If $E \neq \emptyset$, then (71.2) fails in some cases when cosupp J contains some irreducible components of $Y \cap E$. Most likely, this can also be fixed with relatively minor changes, but I do not know how.

Note also that (71.2) would not make sense for any marking different from $m = 1$. Indeed, the ideal I contains the local equations of Y, thus it has order 1. Thus $\mathcal{BMO}_m(X, I, m, \emptyset)$ is the identity for any $m \geq 2$.

72 (Proof of (69) & (71.2) $\Rightarrow$ (35)). The only tricky point is that in (35) E is a usual divisor but (68) assumes that the index set of E is ordered. We can order the index set somehow, so the existence of a principalization is not a problem. However, if we want functoriality, then we should not introduce arbitrary choices in the process.

If, by chance, the irreducible components of E are disjoint, then we can just declare that E is a single divisor, since in (68) we allow the components of E to be reducible. Next we show how to achieve this by some preliminary blow-ups.

Let $E_1, \ldots, E_k$ be the irreducible components of E. We make the E_i disjoint in $k - 1$ steps.

First, let $Z_0 \subset X_0 = X$ be the subset where all of the $E_1, \ldots, E_k$ intersect. Let $\pi_0 : X_1 \to X_0$ be the blow-up of Z_0 with exceptional divisor F^1. Note that the $(\pi_0)_*^{-1} E_1, \ldots, (\pi_0)_*^{-1} E_k$ do not have any k-fold intersections.

Next let $Z_1 \subset X_1$ be the subset where $k - 1$ of the $(\pi_0)_*^{-1} E_i$ intersect. Z_1 is smooth since the $(\pi_0)_*^{-1} E_i$ do not have any k-fold intersections. Let $\pi_1 : X_2 \to X_1$ be the blow-up of Z_1 with exceptional divisor F^2. Note that the $(\pi_0\pi_1)_*^{-1} E_i$ do not have any $(k-1)$-fold intersections.

Next let $Z_2 \subset X_2$ be the subset where $k - 2$ of the $(\pi_0\pi_1)_*^{-1} E_i$ intersect, and so on.

After $(k-1)$-steps we get rid of all pairwise intersections as well. The end result is $\pi : X' \to X$ such that $E^0 := \pi_*^{-1}(E_1 + \cdots + E_k)$ is a smooth divisor. Let $E^1, \ldots, E^{k-1}$ denote the birational transforms of $F^1, \ldots, F^{k-1}$.

Thus $(X', \pi^* I, \sum_{i=0}^{k-1} E^i)$ satisfies the assumptions of (68).

(It may seem natural to start with dim X-fold intersections instead of k-fold intersections. We want functoriality with respect to all smooth morphisms, so we should not use the dimension of X in constructing the resolution process. However, ultimately the difference is only in some empty blow ups, and we can forget about those at the end.)

The rest is straightforward. Construct

$$\Pi_{(X,I,E)} : X_r \to \cdots \to X_s = X' \to \cdots \to X$$

by composing $\mathcal{B}\mathcal{M}\mathcal{O}_1(X', \pi^*I, 1, \sum_{i=0}^{k-1} E^i)$ with $X' \to X$. By construction, $\Pi_{(X,I,E)}^* I = I_r \cdot \mathcal{O}_{X_r}(F)$ for some effective divisor F supported on the total transform of $\sum_{i=0}^{k-1} E^i$. Here $I_r = \mathcal{O}_{X_r}$ since max-ord $I_r < 1$ and F is a simple normal crossing divisor. Therefore $\Pi_{(X,I,E)}^* I$ is a monomial ideal which can be written down explicitly as follows.

Let $F_j \subset X_{j+1}$ denote the exceptional divisor of the jth step in the above smooth blow-up sequence for $\Pi_{(X,I,E)} : X_r \to X$. Then

$$\Pi_{(X,I,E)}^* I = \mathcal{O}_{X_r}\big(-\sum_{j=s}^{r} \Pi_{r,j+1}^* F_j\big),$$

where $\Pi_{r,j+1} : X_r \to X_{j+1}$ is the corresponding composite of blow-ups.

The functoriality properties required in (35) follow from the corresponding functoriality properties in (69) and from (71.2).   □

7. BIRATIONAL TRANSFORM OF DERIVATIVES

Definition 73 (Derivative of an ideal sheaf)**.** On a smooth scheme X over a field k, let Der_X denote the sheaf of derivations $\mathcal{O}_X \to \mathcal{O}_X$. If $x_1, \ldots, x_n$ are local coordinates at a point $p \in X$, then the derivations $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are local generators of Der_X. Derivation gives a k-bilinear map

$$\mathrm{Der}_X \times \mathcal{O}_X \to \mathcal{O}_X.$$

Let $I \subset \mathcal{O}_X$ be an ideal sheaf. Its *first derivative* is the ideal sheaf $D(I)$ generated by all derivatives of elements of I. That is,

$$D(I) := \big(\mathrm{im}[\mathrm{Der}_X \times I \to \mathcal{O}_X]\big). \tag{73.1}$$

Note that $I \subset D(I)$, as shown by the formula

$$f = \frac{\partial(xf)}{\partial x} - x\frac{\partial f}{\partial x}.$$

In terms of generators we can write $D(I)$ as

$$D(f_1, \ldots, f_s) = \Big(f_i, \frac{\partial f_i}{\partial x_j} : 1 \le i \le s, 1 \le j \le n\Big).$$

Higher derivatives are defined inductively by

$$D^{r+1}(I) := D\big(D^r(I)\big). \tag{73.2}$$

(Note that $D^r(I)$ contains all rth partial derivatives of elements of I, but over general rings it is bigger; try second derivatives over $\mathbb{Z}[x]$. Over characteristic zero fields, they are actually equal, as one can see using formulas like

$$\frac{\partial f}{\partial y} = \frac{\partial^2(xf)}{\partial y \partial x} - x\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad 2\frac{\partial f}{\partial x} = \frac{\partial^2(xf)}{\partial x^2} - x\frac{\partial^2 f}{\partial x^2}.$$

The inductive definition is easier to work with.)

If max-ord $I \le m$, then $D^m(I) = \mathcal{O}_X$, and thus the $D^r(I)$ give an ascending chain of ideal sheaves

$$I \subset D(I) \subset D^2(I) \subset \cdots \subset D^m(I) = \mathcal{O}_X.$$

This is, however, not the right way to look at derivatives. Since differentiating a function r times is expected to reduce its order by r, we define the derivative of a marked ideal by

$$D^r(I, m) := \big(D^r(I), m - r\big) \quad \text{for } r \leq m. \tag{73.3}$$

Before we can usefully compare the ideal I and its higher derivatives, we have to correct for the difference in their markings.

Higher derivatives have the usual properties.

Lemma 74. *Let the notation be as above. Then*

(1) $D^r(D^s(I)) = D^{r+s}(I)$,
(2) $D^r(I \cdot J) \subset \sum_{i=0}^r D^i(I) \cdot D^{r-i}(J)$ *(product rule),*
(3) $\operatorname{cosupp}(I, m) = \operatorname{cosupp}(D^r(I), m - r)$ *for $r < m$ (char. 0 only!),*
(4) *if $f : Y \to X$ is smooth, then $D(f^*I) = f^*(D(I))$,*
(5) $D(\hat{I}) = \widehat{D(I)}$, *where ^ denotes completion (55).*   □

74.6 (Aside about positive characteristic). The above definition of higher derivatives is "correct" only in characteristic zero. In general, one should use the *Hasse-Dieudonné derivatives,* which are essentially given by

$$\frac{1}{r_1! \cdots r_n!} \cdot \frac{\partial^{\sum r_i}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}.$$

These operators then have other problems. One of the main difficulties of resolution in positive characteristic is a lack of good replacement for higher derivatives.

75 (Birational transform of derivatives). Let X be a smooth variety, $Z \subset X$ a smooth subvariety and $\pi : B_Z X \to X$ the blow-up with exceptional divisor $F \subset B_Z X$. Let (I, m) be a marked ideal sheaf on X such that $m \leq \operatorname{ord}_Z I$. Choose local coordinates $(x_1, \ldots, x_n)$ such that $Z = (x_1 = \cdots = x_r = 0)$. Then

$$y_1 = \tfrac{x_1}{x_r}, \ldots, y_{r-1} = \tfrac{x_{r-1}}{x_r}, \ y_r = x_r, \ldots, y_n = x_n$$

are local coordinates on a chart of $B_Z X$. Let us compute the derivatives of

$$\pi_*^{-1}\big(f(x_1, \ldots, x_n), m\big) = \big(y_r^{-m} f(y_1 y_r, \ldots, y_{r-1} y_r, y_r, \ldots, y_n), m\big),$$

defined in (60.3). The easy formulas are

$$\begin{aligned}
\tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) &= \pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) \quad \text{for } j < r, \\
\tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) &= \tfrac{1}{y_r} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) \quad \text{for } j > r,
\end{aligned}$$

and a more complicated one using the chain rule for $j = r$:

$$\begin{aligned}
\tfrac{\partial}{\partial y_r} \pi_*^{-1}(f, m) &= \tfrac{y_i}{y_r} \sum_{i<r} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_i} f, m - 1\big) + \tfrac{1}{y_r} \pi_*^{-1}\big(\tfrac{\partial}{\partial x_r} f, m - 1\big) \\
&\quad + \big(\tfrac{-m}{y_r}, -1\big) \cdot \pi_*^{-1}(f, m),
\end{aligned}$$

where, as in (59), multiplying by $\big(\tfrac{-m}{y_r}, -1\big)$ means multiplying the function by $\tfrac{-m}{y_r}$ and lowering the marking by 1.

These can be rearranged to

$$\pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) = \tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) \quad \text{for } j < r, \tag{75.1}$$

$$\pi_*^{-1}\big(\tfrac{\partial}{\partial x_j} f, m - 1\big) = y_r \tfrac{\partial}{\partial y_j} \pi_*^{-1}(f, m) \quad \text{for } j > r, \tag{75.2}$$

$$\begin{aligned}
\pi_*^{-1}\big(\tfrac{\partial}{\partial x_r} f, m - 1\big) &= y_r \tfrac{\partial}{\partial y_r} \pi_*^{-1}(f, m) - y_r \sum_{i<r} \tfrac{\partial}{\partial y_i} \pi_*^{-1}(f, m) \\
&\quad + (m, -1) \cdot \pi_*^{-1}(f, m).
\end{aligned} \tag{75.3}$$

For later purposes, also note the following version of (75.1):

$$\pi_*^{-1}\left(x_j\tfrac{\partial}{\partial x_j}f, m-1\right) \;\;=\;\; y_r y_j \tfrac{\partial}{\partial y_j}\pi_*^{-1}(f,m) \quad \text{for } j < r. \qquad (75.4)$$

Observe that the right-hand sides of these equations are in $D(\pi_*^{-1}(f,m))$. Thus we have proved the following elementary but important statement.

Theorem 76. *Let (I,m) be a marked ideal and $\Pi : X_r \to X$ the composite of a smooth blow-up sequence of order $\geq m$ starting with (X,I,m). Then*

$$\Pi_*^{-1}\left(D^j(I,m)\right) \subset D^j\left(\Pi_*^{-1}(I,m)\right) \quad \text{for every } j \geq 0.$$

Proof. For $j = 1$ and for one blow-up this is what the above formulas (75.1–3) say. The rest follows by induction on j and on the number of blow-ups. $\qquad\square$

Corollary 77. *Let*

$$\Pi : (X_r, I_r, m) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, m) \xrightarrow{\pi_0} (X_0, I_0, m)$$

be a smooth blow-up sequence of order $\geq m$ starting with (X,I,m).
Fix $j \leq m$, and define inductively the ideal sheaves J_i by

$$J_0 := D^j(I) \quad and \quad (J_{i+1}, m-j) := (\pi_i)_*^{-1}(J_i, m-j).$$

Then, $J_i \subset D^j(I_i)$ for every i, and we get a smooth blow-up sequence of order $\geq m-j$ starting with $(X, D^j(I), m-j)$

$$\Pi : (X_r, J_r, m-j) \xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, m-j) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, J_1, m-j) \xrightarrow{\pi_0} (X_0, J_0, m-j).$$

Proof. We need to check that for every $i < r$ the inequality $\mathrm{ord}_{Z_i} J_i \geq m-j$ holds, where $Z_i \subset X_i$ is the center of the blow-up $\pi_i : X_{i+1} \to X_i$. If $\Pi_i : X_i \to X$ is the composition, then

$$J_i = (\Pi_i)_*^{-1}(D^j I, m-j) \subset D^j\left((\Pi_i)_*^{-1}(I,m)\right) = D^j(I_i, m),$$

where the containment in the middle follows from (76). By assumption $\mathrm{ord}_{Z_i} I_i \geq m$, and thus $\mathrm{ord}_{Z_i} D^j(I_i) \geq m-j$ by (74.3). $\qquad\square$

8. Maximal contact and going down

Definition 78. Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\,I$. A smooth hypersurface $H \subset X$ is called a hypersurface of *maximal contact* if the following holds.

For every open set $X^0 \subset X$ and for every smooth blow-up sequence of order m starting with $(X^0, I^0 := I|_{X^0})$,

$$\Pi : (X_r^0, I_r^0) \xrightarrow{\pi_{r-1}} (X_{r-1}^0, I_{r-1}^0) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1^0, I_1^0) \xrightarrow{\pi_0} (X_0^0, I_0^0),$$

the center of every blow-up $Z_i^0 \subset X_i^0$ is contained in the birational transform $H_i^0 \subset X_i^0$ of $H^0 := H \cap X^0$. This implies that

$$\Pi|_{H_r^0} : (H_r^0, I_r|_{H_r^0}, m) \xrightarrow{\pi_{r-1}} (H_{r-1}^0, I_{r-1}|_{H_{r-1}^0}, m) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (H_1^0, I_1|_{H_1^0}, m) \xrightarrow{\pi_0} (H_0^0, I_0|_{H_0^0}, m)$$

is a smooth blow-up sequence of order $\geq m$ starting with $(H^0, I|_{H^0}, m)$.
Being a hypersurface of maximal contact is a local property.

For now we ignore the divisorial part E of a triple (X, I, E) since we cannot guarantee that $E|_H$ is also a simple normal crossing divisor.

Definition 79. Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. The *maximal contact ideal* of I is

$$MC(I) := D^{m-1}(I).$$

Note that $MC(I)$ has order 1 at $x \in X$ if $\text{ord}_x I = m$ and order 0 if $\text{ord}_x I < m$. Thus

$$\text{cosupp}\, MC(I) = \text{cosupp}(I, m).$$

Theorem 80 (Maximal contact). *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Let L be a line bundle on X and $h \in H^0(X, L \otimes MC(I))$ a section with zero divisor $H := (h = 0)$.*

(1) *If H is smooth and $I|_H \neq 0$, then H is a hypersurface of maximal contact.*
(2) *Every $x \in X$ has an open neighborhood $x \in U_x \subset X$ and $h_x \in H^0(U_x, L \otimes MC(I))$ such that $H_x := (h_x = 0) \subset U_x$ is smooth.*

Proof. Being a hypersurface of maximal contact is a local question, and thus we may assume that $L = \mathcal{O}_X$. Let

$$\Pi : (X_r, I_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, I_1) \xrightarrow{\pi_0} (X_0, I_0)$$

be a smooth blow-up sequence of order m starting with (X, I), where π_i is the blow-up of $Z_i \subset X_i$.

Applying (77) for $j = m - 1$, we obtain a smooth blow-up sequence of order ≥ 1 starting with $(X, J_0 := MC(I), 1)$:

$$\Pi : (X_r, J_r, 1) \xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, 1) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} (X_1, J_1, 1) \xrightarrow{\pi_0} (X_0, J_0, 1).$$

Let $H_i := (\Pi_i)_*^{-1} H \subset X_i$ denote the birational transform of $H \subset X$. Since $\mathcal{O}_{X_0}(-H_0) \subset J_0$ and H_0 is smooth, we see that $\mathcal{O}_{X_i}(-H_i) \subset J_i$ for every i. By assumption $\text{ord}_{Z_i} I_i \geq m$. Thus, using (74.3) and (77) we get that

$$\text{ord}_{Z_i} J_i \geq \text{ord}_{Z_i} MC(I_i) \geq 1$$

and hence also $\text{ord}_{Z_i} H_i \geq 1$. Thus $Z_i \subset H_i$ for every i, and so H is a hypersurface of maximal contact.

To see the second claim, pick $x \in X$ such that $\text{ord}_x I = m$. Then $\text{ord}_x MC(I) = 1$ by (74.3). Thus there is a local section of $MC(I)$ that has order 1 at x, and so its zero divisor is smooth in a neighborhood of x. $\qquad\square$

Aside 81. A section $h \in MC(I)$ such that $H = (h = 0)$ is smooth always exists locally but usually not globally, not even if we tensor I by a very ample line bundle L. By the Bertini-type theorem of [Kol97, 4.4], the best one can achieve globally is that H has cA-type singularities. (These are given by local equations $x_1 x_2 + $ (other terms) $= 0$.)

The above results say that every smooth blow-up sequence of order m starting with (X, I) can be seen as a smooth blow-up sequence starting with $(H, I|_H, m)$.

An important remaining problem is that not every smooth blow-up sequence starting with $(H, I|_H, m)$ corresponds to a smooth blow-up sequence of order m starting with (X, I), and thus we cannot yet construct an order reduction of (X, I) from an order reduction of $(H, I|_H, m)$.

Here are some examples that show what can go wrong.

Example 82. Let $I = (xy - z^n)$. Then $\mathrm{ord}_0 I = 2$ and $D(I) = (x, y, z^{n-1})$. $H := (x = 0)$ is a surface of maximal contact, and

$$(H, I|_H) \cong \left(\mathbb{A}^2_{y,z}, (z^n)\right).$$

Thus $(H, I|_H)$ shows a 1-dimensional singular locus of order n, whereas we have an isolated singular point of order 2. The same happens if we use $(y = 0)$ as a surface of maximal contact.

In this case we do better if we use a general surface of maximal contact. Indeed, for $H_g := (x - y = 0)$,

$$(H_g, I|_{H_g}) \cong \left(\mathbb{A}^2_{x,z}, (x^2 - z^n)\right),$$

and we get an equivalence between smooth blow-up sequences of order 2 starting with $(\mathbb{A}^3, (xy - z^n))$ and smooth blow-up sequences of order ≥ 2 starting with $(\mathbb{A}^2, (x^2 - z^n), 2)$.

In some cases, even the general hypersurface of maximal contact fails to produce an equivalence. There are no problems on H itself, but difficulties appear after blow-ups.

Let $I = (x^3 + xy^5 + z^4)$. A general surface of maximal contact is

$$H := (x + u_1 xy^3 + u_2 y^4 + u_3 z^2 = 0), \quad \text{where the } u_i \text{ are units.}$$

Let us compute two blow-ups given by $x_1 = x/y, y_1 = y, z_1 = z/y$ and $x_2 = x_1/y_1, y_2 = y_1, z_2 = z_1/y_1$. We get the birational transforms

$$
\begin{array}{ll}
x^3 + xy^5 + z^4 & x + u_1 xy^3 + u_2 y^4 + u_3 z^2 \\
x_1^3 + x_1 y_1^3 + y_1 z_1^4 & x_1 + u_1 x_1 y_1^3 + u_2 y_1^3 + u_3 y_1 z_1^2 \\
x_2^3 + x_2 y_2 + y_2^2 z_2^4 & x_2 + u_1 x_2 y_2^3 + u_2 y_2^2 + u_3 y_2^2 z_2^2.
\end{array}
$$

The second birational transform of the ideal has order 2 on this chart. However, its restriction to the birational transform H_2 of H still has order 3 since we can use the equation of H_2 to eliminate x_2 by the substitution

$$x_2 = -y_2^2 (u_2 + u_3 z_2^2)(1 + u_1 y_2^3)^{-1}$$

to obtain that $I_2|_{H_2} \subset (y_2^3, y_2^2 z_2^4)$.

9. RESTRICTION OF DERIVATIVES AND GOING UP

In general, neither the order of an ideal nor its derivative ideal commute with restrictions to smooth hypersurfaces. For instance, if $I = (x^2 + xy + z^3)$ and $S = (x = 0)$ then $\mathrm{ord}_0 I = 2$ but $\mathrm{ord}_0(I|_S) = 3$ and $(DI)|_S = (y, z^2)$ but $D(I|_S) = (z^2)$. It is easy to see that

$$\mathrm{ord}_p I \leq \mathrm{ord}_p(I|_S) \quad \text{and} \quad (DI)|_S \supset D(I|_S),$$

but neither is an equality. The notion of D-balanced ideals provides a solution to the first of these problems and a partial remedy to the second.

Definition 83. As in (52), an ideal I with $m = \text{max-ord}\, I$ is called D-*balanced* if

$$\left(D^i I\right)^m \subset I^{m-i} \quad \forall\, i < m.$$

If I is D-balanced, then at every point it has order either m or 0. Indeed, if $\mathrm{ord}_p I < m$ then $(D^{m-1}I)_p = \mathcal{O}_{p,X}$, thus I^{m-1} and I both contain a unit at p. In particular, $\mathrm{cosupp}(I, m) = \mathrm{cosupp}\, I$, hence the maximal order commutes with restrictions.

We can reformulate this observation as follows. If I is D-balanced, then any smooth blow-up of order $\geq m$ for $I|_S$ corresponds to a smooth blow-up of order $\geq m$ for I. We would like a similar statement not just for one blow-up, but for all blow-up sequences.

Theorem 84 (Going-up property of D-balanced ideals)**.** *Let X be a smooth variety and I a D-balanced sheaf of ideals with $m = \text{max-ord}\, I$. Let $S \subset X$ be any smooth hypersurface such that $S \not\subset \text{cosupp}(I, m)$ and*

$$\Pi^S : (S_r, J_r, m) \xrightarrow{\pi^S_{r-1}} (S_{r-1}, J_{r-1}, m) \xrightarrow{\pi^S_{r-2}} \cdots$$
$$\xrightarrow{\pi^S_1} (S_1, J_1, m) \xrightarrow{\pi^S_0} (S_0, J_0, m) = (S, I|_S, m)$$

be a smooth blow-up sequence of order $\geq m$, where π^S_i is the blow-up of $Z_i \subset S_i$. Then the pushed-forward sequence (30)

$$\Pi : (X_r, I_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1) \xrightarrow{\pi_0} (X_0, I_0) = (X, I)$$

is a smooth blow-up sequence of order m, where π_i is the blow-up of $Z_i \subset S_i \subset X_i$.

Corollary 85 (Going up and down)**.** *Let X be a smooth variety, $I \subset \mathcal{O}_X$ a D-balanced ideal sheaf with $m = \text{max-ord}\, I$ and E a divisor with simple normal crossings. Let $H \subset X$ be a smooth hypersurface of maximal contact such that $E + H$ is also a divisor with simple normal crossings and no irreducible component of H is contained in $\text{cosupp}(I, m)$.*

Then pushing forward (30) from H to X is a one-to-one correspondence between

(1) *smooth blow-up sequences of order $\geq m$ starting with the triple $(H, I|_H, m, E|_H)$, and*

(2) *smooth blow-up sequences of order m starting with (X, I, E).*

Proof. This follows from (84) and (80), except for the role played by E.

Adding E to (X, I) (resp., to $(H, m, I|_H)$) means that now we can use only blow-ups whose centers are in simple normal crossing with E (resp., $E|_H$) and their total transforms. Since $E|_H$ is again a divisor with simple normal crossings, this poses the same restriction on order reduction for (X, I, E) as on order reduction for $(H, I|_H, m, E|_H)$. $\qquad\square$

86 (First attempt to prove (84))**.** We already noted that we are ok for the first blow-up. Let us see what happens with pushing forward the second blow-up $\pi^S_1 : S_2 \to S_1$. By assumption $\text{ord}_{Z_1}(\pi^S_0)^{-1}_*(I|_S, m) \geq m$. Can we conclude from this that $\text{ord}_{Z_1}(\pi_0)^{-1}_*(I, m) \geq m$? In other words, is

$$S_1 \cap \text{cosupp}\big((\pi_0)^{-1}_*(I, m)\big) = \text{cosupp}\big((\pi^S_0)^{-1}_*(I|_S, m)\big)?$$

Since the birational transform commutes with restrictions, this indeed holds if the birational transform $(\pi_0)^{-1}_*(I, m)$ is again D-balanced. By assumption $(D^i I)^m \subset I^{m-i}$ and so

$$\big((\pi_0)^{-1}_*(D^i I, m - i)\big)^m \subset \big((\pi_0)^{-1}_*(I, m)\big)^{m-i}.$$

Unfortunately, when we interchange $(\pi_0)^{-1}_*$ and D^i on the left-hand side, the inequality in (76) goes the wrong way and indeed, in general the birational transform is not D-balanced.

Looking at the formulas (75.1–3), we see that taking birational transform commutes with some derivatives but not with others.

In order to exploit this, we introduce logarithmic derivatives. This notion enables us to separate the "good" directions from the "bad" ones.

Example 86.1. Check that (x^2, xy^m, y^{m+1}) is D-balanced. After blowing up the origin, one of the charts gives $(x_1^2, x_1 y_1^{m-1}, y_1^{m-1})$, which is not D-balanced.

87 (Logarithmic derivatives)**.** Let X be a smooth variety and $S \subset X$ a smooth subvariety. For simplicity, we assume that S is a hypersurface. At a point $p \in S$ pick local coordinates $x_1, \ldots, x_n$ such that $S = (x_1 = 0)$. If f is any function, then

$$\frac{\partial f}{\partial x_i}\Big|_S = \frac{\partial\big(f|_S\big)}{\partial x_i} \quad \text{for } i > 1,$$

but $\partial(f|_S)/\partial x_1$ does not even make sense. Therefore, we would like to decompose $D(f)$ into two parts:

- $\partial f/\partial x_i$ for $i > 1$ (these commute with restriction to S), and
- $\partial f/\partial x_1$ (which does not).

Such a decomposition is, however, not coordinate invariant. The best one can do is the following.

Let $\mathrm{Der}_X(-\log S) \subset \mathrm{Der}_X$ be the largest subsheaf that maps $\mathcal{O}_X(-S)$ into itself by derivations. It is called the sheaf of *logarithmic derivations* along S. In the above local coordinates we can write

$$\mathrm{Der}_X(-\log S) = \Big(x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\Big).$$

For an ideal sheaf I set

$$
\begin{aligned}
D(-\log S)(I) &:= \big(\mathrm{im}[\mathrm{Der}_X(-\log S) \times I \to \mathcal{O}_X]\big) \quad \text{and} \\
D^{r+1}(-\log S)(I) &:= D(-\log S)\big(D^r(-\log S)(I)\big) \quad \text{for } r \geq 1.
\end{aligned}
$$

We need three properties of log derivations.

First, log derivations behave well with respect to restriction to S:

$$\big(D^r(-\log S)(I)\big)\big|_S = D^r(I|_S). \tag{87.1}$$

Second, one can filter the sheaf $D^s(I)$ by subsheaves

$$D^s(-\log S)(I) \subset D^{s-1}(-\log S)\big(D(I)\big) \subset \cdots \subset D^s(I).$$

There are no well-defined complements, but in local coordinates $x_1, \ldots, x_n$ we can write

$$D^s(I) = D^s(-\log S)(I) + D^{s-1}(-\log S)\Big(\frac{\partial I}{\partial x_1}\Big) + \cdots + \Big(\frac{\partial^s I}{\partial x_1^s}\Big), \tag{87.2}$$

and the first $j + 1$ summands span $D^{s-j}(-\log S)(D^j(I))$.

Third, under the assumptions of (84), we get a logarithmic version of (76):

$$\Pi_*^{-1}\Big(D^j(-\log S_r)(I, m)\Big) \subset D^j(-\log S)\Big(\Pi_*^{-1}(I, m)\Big), \tag{87.3}$$

which is proved the same way using (75.4).

We can now formulate the next result, which can be viewed as a way to reverse the inclusion in (76).

Theorem 88. *Consider a smooth blow-up sequence of order $\geq m$:*

$$\Pi : (X_r, I_r, m) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, m) \xrightarrow{\pi_0} (X_0, I_0, m) = (X, I, m).$$

Let $S \subset X$ be a smooth hypersurface and $S_i \subset X_i$ its birational transforms. Assume that each blow-up center Z_i is contained in S_i. Then

$$D^s \Pi_*^{-1}(I, m) = \sum_{j=0}^{s} D^{s-j}(-\log S_r) \Pi_*^{-1}(D^j I, m - j). \qquad (88.1)$$

Proof. Using (76) we obtain that

$$\begin{aligned}
D^{s-j}(-\log S_r)\Pi_*^{-1}(D^j I, m - j) &\subset D^{s-j}\Pi_*^{-1}(D^j I, m - j) \\
&\subset D^{s-j}D^j(\Pi_*^{-1}I, m) = D^s\Pi_*^{-1}(I, m),
\end{aligned}$$

and thus the right-hand side of (88.1) is contained in the left-hand side.

Next let us check the reverse inclusion in (88.1) for one blow-up. The question is local on X, and so choose coordinates $x_1, \ldots, x_n$ such that $S = (x_1 = 0)$ and the center of the blow-up π is $(x_1 = \cdots = x_r = 0)$. We have a typical local chart

$$y_1 = \frac{x_1}{x_r}, \ldots, y_{r-1} = \frac{x_{r-1}}{x_r}, \ y_r = x_r, \ldots, y_n = x_n,$$

and $S_1 = (y_1 = 0)$ is the birational transform of S. Note that the blow-up is covered by r different charts, but only $r - 1$ of these can be written in the above forms, where x_r is different from x_1. These $r - 1$ charts, however, completely cover S_1.

Applying (87.2) to $\pi_*^{-1}(I, m)$ we obtain that

$$D^s \pi_*^{-1}(I, m) = \sum_{j=0}^{s} D^{s-j}(-\log S_1)\left(\frac{\partial^j \pi_*^{-1}(I,m)}{\partial y_1^j}\right).$$

Although usually differentiation does not commute with birational transforms, by (75.1) it does so for $\partial/\partial x_1$ and $\partial/\partial y_1$. So we can rewrite the above formula as

$$\begin{aligned}
D^s \pi_*^{-1}(I, m) &= \sum_{j=0}^{s} D^{s-j}(-\log S_1)\pi_*^{-1}\left(\frac{\partial^j (I,m)}{\partial x_1^j}\right) \\
&\subset \sum_{j=0}^{s} D^{s-j}(-\log S_1)\pi_*^{-1}(D^j I, m - j),
\end{aligned} \qquad (88.2)$$

where the inclusion is clear. As noted above, the right-hand side of (88.2) is contained in the left-hand side, and hence they are equal. This proves (88) for one blow-up.

In the general case, we use induction on the number of blow-ups. We factor $\Pi : X_r \to X$ as the composite of $\pi_{r-1} : X_r \to X_{r-1}$ and $\Pi_{r-1} : X_{r-1} \to X$. Use (88) for π_{r-1} to get that

$$\begin{aligned}
D^s \Pi_*^{-1}(I, m) &= D^s(\pi_{r-1})_*^{-1}(\Pi_{r-1})_*^{-1}(I, m) \\
&= \sum_{j=0}^{s} D^{s-j}(-\log S_r)(\pi_{r-1})_*^{-1}D^j(\Pi_{r-1})_*^{-1}(I, m).
\end{aligned} \qquad (88.3)$$

By induction (88) holds for Π_{r-1} and $s = j$, thus

$$D^j (\Pi_{r-1})_*^{-1}(I, m) = \sum_{\ell=0}^{j} D^{j-\ell}(-\log S_{r-1})(\Pi_{r-1})_*^{-1}(D^\ell I, m - \ell).$$

By (87.3), we can interchange $(\pi_{r-1})_*^{-1}$ and $D^{j-\ell}(-\log S_{r-1})$, and so

$$(\pi_{r-1})_*^{-1} D^j (\Pi_{r-1})_*^{-1}(I, m)$$
$$= (\pi_{r-1})_*^{-1} \sum_{\ell=0}^j D^{j-\ell}(-\log S_{r-1})(\Pi_{r-1})_*^{-1}\big(D^\ell I, m - \ell\big)$$
$$\subset \sum_{\ell=0}^j D^{j-\ell}(-\log S_r)(\pi_{r-1})_*^{-1}(\Pi_{r-1})_*^{-1}\big(D^\ell I, m - \ell\big)$$
$$= \sum_{\ell=0}^j D^{j-\ell}(-\log S_r)\Pi_*^{-1}\big(D^\ell I, m - \ell\big).$$

Substituting into (88.3), we obtain the desired result:

$$D^s \Pi_*^{-1}(I, m)$$
$$\subset \sum_{j=0}^s D^{s-j}(-\log S_r) \sum_{\ell=0}^j D^{j-\ell}(-\log S_r)\Pi_*^{-1}\big(D^\ell I, m - \ell\big)$$
$$= \sum_{\ell=0}^s D^{s-\ell}(-\log S_r)\Pi_*^{-1}\big(D^\ell I, m - \ell\big). \quad \square$$

Corollary 89. *Let the notation and assumptions be as in (88). Then*

$$S_r \cap \mathrm{cosupp}\big(\Pi_*^{-1}(I, m)\big) = \bigcap_{j=0}^{m-1} \mathrm{cosupp}(\Pi|_{S_r})_*^{-1}\big((D^j I)|_S, m - j\big).$$

Proof. Restricting (88.1) to S_r and using (62) and (87.1) we get that

$$\big(D^s \Pi_*^{-1}(I, m)\big)|_{S_r} = \sum_{j=0}^s D^{s-j}(\Pi|_{S_r})_*^{-1}\big((D^j I)|_S, m - j\big). \qquad (89.1)$$

Set $s = m-1$ and take cosupports. Since $D^{m-1}\Pi_*^{-1}(I, m)$ has order 1, its cosupport commutes with restrictions, so the left-hand side of (89.1) becomes

$$\mathrm{cosupp}\Big(\big(D^{m-1}\Pi_*^{-1}(I, m)\big)|_{S_r}\Big) = \mathrm{cosupp}\big(D^{m-1}\Pi_*^{-1}(I, m)\big) \cap S_r$$
$$= \mathrm{cosupp}\big(\Pi_*^{-1}(I, m)\big) \cap S_r, \qquad (89.2)$$

where the second equality follows from (74.3).

On the right-hand side of (89.1) use (59.4) to obtain that

$$\mathrm{cosupp}\Big(\sum_{j=0}^{m-1} D^{m-1-j}(\Pi|_{S_r})_*^{-1}\big((D^j I)|_S, m - j\big)\Big)$$
$$= \bigcap_{j=0}^{m-1} \mathrm{cosupp}\Big(D^{m-1-j}(\Pi|_{S_r})_*^{-1}\big((D^j I)|_S, m - j\big)\Big) \qquad (89.3)$$
$$= \bigcap_{j=0}^{m-1} \mathrm{cosupp}(\Pi|_{S_r})_*^{-1}\big((D^j I)|_S, m - j\big).$$

The last lines of (89.2) and (89.3) are thus equal. $\square$

90 (Proof of (84)). By induction, assume that this already holds for blow-up sequences of length $< r$. We need to show that the last blow-up also has order $\geq m$, or, equivalently, $\mathrm{cosupp}(J_{r-1}, m) \subset \mathrm{cosupp}(I_{r-1}, m)$.

Using first (89) for $\Pi_{r-1} : X_{r-1} \to X$, then the D-balanced property in line 2, we obtain that

$$S_{r-1} \cap \mathrm{cosupp}(I_{r-1}, m) = \bigcap_{j=0}^{m-1} \mathrm{cosupp}(\Pi_{r-1}^S)_*^{-1}\big((D^j I)|_S, m - j\big)$$
$$= \bigcap_{j=0}^{m-1} \mathrm{cosupp}(\Pi_{r-1}^S)_*^{-1}\big((D^j I)^m|_S, m(m - j)\big)$$
$$\supset \bigcap_{j=0}^{m-1} \mathrm{cosupp}(\Pi_{r-1}^S)_*^{-1}\big(I^{m-j}|_S, m(m - j)\big)$$
$$= \bigcap_{j=0}^{m-1} \mathrm{cosupp}(\Pi_{r-1}^S)_*^{-1}\big(I|_S, m\big)$$
$$= \mathrm{cosupp}(\Pi_{r-1}^S)_*^{-1}\big(J_0, m\big)$$
$$= \mathrm{cosupp}(J_{r-1}, m). \quad \square$$

10. Uniqueness of maximal contact

Given (X, I, E), let $j : H \hookrightarrow X$ and $j' : H' \hookrightarrow X$ be two hypersurfaces of maximal contact. By (85) we can construct smooth blow-up sequences for (X, I, E) from $(H, I|_H, m, E_H)$ and also from $(H', I|_{H'}, m, E_{H'})$. We need to guarantee that we get the same blow-up sequences.

Assume that there is an automorphism ϕ of X such that $\phi^* I = I$ and $\phi^{-1}(E + H') = E + H$. Then $(H, I_H, m, E_H) = \phi^*(H', I|_{H'}, m, E_{H'})$, thus if $\mathcal{B}(H', I|_{H'}, m, E_{H'})$ is the smooth blow-up sequence constructed using H', then the "same" construction using H gives

$$\mathcal{B}(H, I_H, m, E_H) = \phi^* \, \mathcal{B}(H', I|_{H'}, m, E_{H'}).$$

Pushing these forward as in (85), we obtain that

$$j_* \, \mathcal{B}(H, I_H, m, E_H) = \phi^* \big(j'_* \, \mathcal{B}(H', I|_{H'}, m, E_{H'}) \big).$$

That is, the blow-up sequences we get from H and H' are isomorphic, but we would like them to be identical.

Let Z_0 (resp., Z'_0) be the center of the first blow-up obtained using H (resp., H'). As above, $\phi^{-1}(Z'_0) = Z_0$. Both Z'_0 and Z_0 are contained in $\mathrm{cosupp}(I, m)$, so if ϕ is the identity on $\mathrm{cosupp}(I, m)$ then $Z'_0 = Z_0$.

The assumption $\phi^* I = I$ implies that ϕ maps $\mathrm{cosupp}(I, m)$ into itself, but it does not imply that ϕ is the identity on $\mathrm{cosupp}(I, m)$. How can we achieve the latter? Let R be a ring, $J \subset R$ an ideal and σ an automorphism of R. It is easy to see that $\sigma(J) = J$ and σ induces the identity automorphism on R/J iff $r - \sigma(r) \in J$ for every $r \in R$.

How should we choose the ideal J in our situation? It turns out that $J = I$ does not work and the ideal sheaf of $\mathrm{cosupp}(I, m)$ behaves badly for blow-ups. An intermediate choice is given by $D^{m-1}(I) = MC(I)$, which works well.

Another twist is that usually X itself has no automorphisms (not even Zariski locally), so we have to work in a formal or étale neighborhood of a point $x \in X$. (See (55) for completions.)

Definition 91. Let X be a smooth variety, $p \in X$ a point, I an ideal sheaf such that $\text{max-ord}\, I = \mathrm{ord}_p I = m$ and $E = E^1 + \cdots + E^s$ a simple normal crossing divisor. Let $H, H' \subset X$ be two hypersurfaces of maximal contact.

We say that H and H' are *formally equivalent at p* with respect to (X, I, E) if there is an automorphism $\phi : \hat{X} \to \hat{X}$ which moves $(X, I, H + E)$ into $(X, I, H' + E)$ and ϕ is close to the identity. That is,

   (1) $\phi(\hat{H}) = \hat{H}'$,
   (2) $\phi^*(\hat{I}) = \hat{I}$,
   (3) $\phi(\hat{E}^i) = \hat{E}^i$ for $i = 1, \ldots, s$, and
   (4) $h - \phi^*(h) \in MC(\hat{I})$ for every $h \in \hat{\mathcal{O}}_{x,X}$.

While this is the important concept, it is somewhat inconvenient to use since we defined resolution, order reduction, and so on for schemes of finite type and not for general schemes like $\hat{X}$.

Even very simple formal automorphisms cannot be realized as algebraic automorphisms on some étale cover. (Check this for the map $x \mapsto \sqrt{x}$, which is a formal automorphism of $(1 \in \hat{\mathbb{C}})$.) Thus we need a slightly modified definition.

We say that H and H' are *étale equivalent* with respect to (X, I, E) if there are étale surjections $\psi, \psi' : U \rightrightarrows X$ such that

(1') $\psi^{-1}(H) = \psi'^{-1}(H')$,

(2') $\psi^*(I) = \psi'^*(I)$,

(3') $\psi^{-1}(E^i) = \psi'^{-1}(E^i)$ for $i = 1, \ldots, s$, and

(4') $\psi^*(h) - \psi'^*(h) \in MC(\psi^*(I))$ for every $h \in \mathcal{O}_X$.

The connection with the formal case comes from noting that ψ is invertible after completion, and then $\phi := \hat{\psi}' \circ \hat{\psi}^{-1} : \hat{X} \to \hat{X}$ is the automorphism we seek.

A key observation of [Wło05] is that for certain ideals I any two smooth hypersurfaces of maximal contact are formal and étale equivalent. Recall (53) that an ideal I is MC-invariant if

$$MC(I) \cdot D(I) \subset I,$$

where $MC(I)$ is the ideal of maximal contacts defined in (51.2). Since taking derivatives commutes with completion (74.5), we see that $\widehat{MC(I)} = MC(\hat{I})$.

Theorem 92 (Uniqueness of maximal contact). *Let X be a smooth variety over a field of characteristic zero, I an MC-invariant ideal sheaf, $m = \text{max-ord}\, I$ and E a simple normal crossing divisor. Let $H, H' \subset X$ be two smooth hypersurfaces of maximal contact for I such that $H + E$ and $H' + E$ both have simple normal crossings.*

Then H and H' are étale equivalent with respect to (X, I, E).

We start with a general result relating automorphisms and derivations of complete local rings. Since derivations are essentially the first order automorphisms, it is reasonable to expect that an ideal is invariant under a subgroup of automorphisms iff it is invariant to first order. We are, however, in an infinite-dimensional setting, so it is safer to work out the details.

Notation 93. Let k be a field of characteristic zero, K/k a finite field extension and $R = K[[x_1, \ldots, x_n]]$ the formal power series ring in n variables with maximal ideal m, viewed as a k-algebra. For $g_1, \ldots, g_n \in m$ the map $g : x_i \mapsto g_i$ extends to an automorphism of $R \Leftrightarrow g : m/m^2 \to m/m^2$ is an isomorphism $\Leftrightarrow$ the linear parts of the g_i are linearly independent.

Let $B \subset m$ be an ideal. For $b_i \in B$ the map $g : x_i \mapsto x_i + b_i$ need not generate an automorphism, but $g : x_i \mapsto x_i + \lambda_i b_i$ gives an automorphism for general $\lambda_i \in k$. We call these *automorphisms of the form* $\mathbf{1} + B$.

Proposition 94. *Let the notation be as above, and let $I \subset R$ be an ideal. The following are equivalent:*

(1) *I is invariant under every automorphism of the form $\mathbf{1} + B$,*

(2) *$B \cdot D(I) \subset I$,*

(3) *$B^j \cdot D^j(I) \subset I$ for every $j \geq 1$.*

Proof. Assume that $B^j \cdot D^j(I) \subset I$ for every $j \geq 1$. Given any $f \in I$, we need to prove that $f(x_1 + b_1, \ldots, x_n + b_n) \in I$. Take the Taylor expansion

$$f(x_1 + b_1, \ldots, x_n + b_n) = f(x_1, \ldots, x_n) + \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} b_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \cdots.$$

For any $s \geq 1$, this gives that

$$f(x_1 + b_1, \ldots, x_n + b_n) \in I + B \cdot D(I) + \cdots + B^s \cdot D^s(I) + m^{s+1} \subset I + m^{s+1}$$

since $B^j \cdot D^j(I) \subset I$ by assumption. Letting s go to infinity, by Krull's intersection theorem (55) we conclude that $f(x_1 + b_1, \ldots, x_n + b_n) \in I$.

Conversely, for any $b \in B$ and general $\lambda_i \in k$, invariance under the automorphism $(x_1, x_2, \ldots, x_n) \mapsto (x_1 + \lambda_i b, x_2, \ldots, x_n)$ gives that

$$f(x_1 + \lambda_i b, x_2, \ldots, x_n) = f(x_1, \ldots, x_n) + \lambda_i b \frac{\partial f}{\partial x_1} + \cdots (\lambda_i b)^s \frac{\partial f^s}{\partial x_1^s} \in I + m^{s+1}.$$

Use s different values $\lambda_1, \ldots, \lambda_s$. Since the Vandermonde determinant (λ_i^j) is invertible, we conclude that

$$b \frac{\partial f}{\partial x_1} \in I + m^{s+1}.$$

Letting s go to infinity, we obtain that $B \cdot D(I) \subset I$.

Finally, we prove by induction that $B^j \cdot D^j(I) \subset I$ for every $j \geq 1$. $B^{j+1} \cdot D^{j+1}(I)$ is generated by elements of the form $b_0 \cdots b_j \cdot D(g)$, where $g \in D^j(I)$. The product rule gives that

$$
\begin{aligned}
b_0 \cdots b_j \cdot D(g) \;&=\; b_0 \cdot D(b_1 \cdots b_j \cdot g) - \sum_{i \geq 1} D(b_i) \cdot (b_0 \cdots \widehat{b_i} \cdots b_j \cdot g) \\
&\in\; B \cdot D\big(B^j \cdot D^j(I)\big) + B^j \cdot D^j(I) \\
&\subset\; B \cdot D(I) + B^j \cdot D^j(I) \subset I,
\end{aligned}
$$

where the entry $\widehat{b_i}$ is omitted from the products. $\qquad\square$

95 (Proof of (92)). Let us start with formal equivalence.

Pick local sections $x_1, x_1' \in MC(I)$ such that $H = (x_1 = 0)$ and $H' = (x_1' = 0)$. Choose other local coordinates $x_2, \ldots, x_{s+1}$ at p such that $E^i = (x_{i+1} = 0)$ for $i = 1, \ldots, s$. For a general choice of $x_{s+2}, \ldots, x_n$, we see that $x_1, x_2, \ldots, x_n$ and $x_1', x_2, \ldots, x_n$ are both local coordinate systems.

If X is a k-variety and the residue field of $p \in X$ is K, then $\widehat{\mathcal{O}}_{p,X} \cong K[[x_1, \ldots, x_n]]$ by (55), so the computations of (94) apply.

Since $x_1 - x_1' \in MC(I)$, the automorphism

$$\phi^*(x_1', x_2, \ldots, x_n) = \big(x_1' + (x_1 - x_1'), x_2, \ldots, x_n\big) = (x_1, x_2, \ldots, x_n)$$

is of the form $\mathbf{1} + MC(I)$. Hence by (94) we conclude that $\phi^*\big(\hat{I}\big) = \hat{I}$. By construction $\phi(\hat{H}) = \hat{H}'$, $\phi(\hat{E}^i) = \hat{E}^i$ and (91.4) is also clear.

In order to go from the formal to the étale case, the key point is to realize the automorphism ϕ on some étale neighborhood. Existence follows from the general approximation theorems of [Art69], but in our case the choice is clear.

Take $X \times X$, and for some $p \in X$ let $x_{11}, x_{12}, \ldots, x_{1n}$ be the corresponding local coordinates on the first factor and $x_{21}', x_{22}, \ldots, x_{2n}$ on the second factor. Set

$$U_1(p) := (x_{11} - x_{21}' = x_{12} - x_{22} = \cdots = x_{1n} - x_{2n} = 0) \subset X \times X.$$

The completion of $U_1(p)$ at (p, p) is the graph of ϕ_p. By shrinking $U_1(p)$, we get $(p, p) \in U_2(p) \subset U_1(p)$ such that both coordinate projections $\psi_p, \psi_p' : U_2(p) \rightrightarrows X$ are étale.

From our previous considerations, we know that (91.1–4) hold after taking completions at (p, p). Thus (91.1′–4′) also hold in an open neighborhood $U(p) \ni (p, p)$ by (55).

The images of finitely many of the $U(p)$ cover X. We can take U to be their disjoint union. $\qquad\square$

In Section 3.12 we use the maximal contact hypersurfaces H, H' to construct blow-up sequences $\mathbf{B}$ and $\mathbf{B}'$ which become isomorphic after pulling back to U. The next result shows that they are the same already on X. That is, our blow-ups do not depend on the choice of a hypersurface of maximal contact.

Definition 96. Let X be a smooth variety over a field of characteristic zero and let

$$\mathbf{B} \;:=\; (X_r, I_r) \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_0} (X_0, I_0) = (X, I), \quad \text{and}$$
$$\mathbf{B}' \;:=\; (X_r', I_r') \xrightarrow{\pi_{r-1}'} \cdots \xrightarrow{\pi_0'} (X_0', I_0') = (X, I)$$

be two blow-up sequences of order $m = \text{max-ord}\, I$. We say that $\mathbf{B}$ and $\mathbf{B}'$ are *étale equivalent* if there are étale surjections $\psi, \psi' : U \rightrightarrows X$ such that

(1) $\psi^*(I) = \psi'^*(I)$,
(2) $\psi^*(h) - \psi'^*(h) \in MC\big(\psi^*(I)\big)$ for every $h \in \mathcal{O}_X$, and
(3) $\psi^*\mathbf{B} = \psi'^*\mathbf{B}'$.

Theorem 97. *Let X be a smooth variety over a field of characteristic zero and I an MC-invariant ideal sheaf. Let $\mathbf{B}$ and $\mathbf{B}'$ be two blow-up sequences of order $m = \text{max-ord}\, I$ which are étale equivalent.*
   Then $\mathbf{B} = \mathbf{B}'$.

Proof. By assumption there are étale surjections $\psi, \psi' : U \rightrightarrows X$ such that $\psi^*\mathbf{B} = \psi'^*\mathbf{B}'$. Let

$$\mathbf{B}^U := (U_r, I_r^U) \xrightarrow{\pi_{r-1}^U} \cdots \xrightarrow{\pi_0^U} (U_0, I_0^U) = (U, \psi^*I = \psi'^*I)$$

be the common pullback. We prove by induction on i the following claims.

(1) $(X_i, I_i) = (X_i', I_i')$.
(2) ψ, ψ' lift to étale surjections $\psi_i, \psi'_i : U_i \rightrightarrows X_i$ such that

$$\text{im}(\psi_i^* - \psi_i'^*) \subset (\Pi_i^U)_*^{-1}\big(MC(I_0^U), 1\big).$$

(3) $Z_{i-1} = Z_{i-1}'$.

For $i = 0$ there is nothing to prove. Let us see how to go from i to $i+1$. Set $W_i = \text{cosupp}(\Pi_i^U)_*^{-1}\big(MC(I_0^U), 1\big)$ and note that $Z_i^U \subset W_i$ by (77). By the inductive assumption (2) $\psi_i|_{W_i} = \psi_i'|_{W_i}$, thus $Z_i = \psi_i(Z_i^U) = \psi_i'(Z_i^U) = Z_i'$. This in turn implies that $X_{i+1} = X_{i+1}'$.

In order to compute the lifting of ψ_i and ψ_i', pick local coordinates $x_1, \ldots, x_n$ on $X_i = X_i'$ such that $Z_i = Z_i' = (x_1 = x_2 = \cdots = x_k = 0)$. By induction,

$$\psi_i'^*(x_j) = \psi_i^*(x_j) - b(i,j) \quad \text{for some } b(i,j) \in (\Pi_i^U)_*^{-1}\big(MC(I^U), 1\big).$$

On the blow-up $\pi_i : X_{i+1} \to X_i$ consider the local chart

$$y_1 = \tfrac{x_1}{x_r}, \ldots, y_{r-1} = \tfrac{x_{r-1}}{x_r}, \ y_r = x_r, \ldots, y_n = x_n.$$

We need to prove that

$$\psi_{i+1}^*(y_j) - \psi_{i+1}'^*(y_j) \in (\Pi_{i+1}^U)_*^{-1}\big(MC(I^U), 1\big)$$

for every j. This is clear if $y_j = x_j$, that is, for $j \geq r$. Next we compute the case when $j < r$.

The $b(i,j)$ vanish along Z_i^U and so $(\pi_i^U)^* b(i,j) = \psi_{i+1}^*(x_r) b(i+1,j)$ for some $b(i+1,j) \in (\Pi_{i+1}^U)_*^{-1}\big(MC(I^U), 1\big)$. Hence, for $j < r$, we obtain that

$$
\begin{aligned}
\psi_{i+1}'^*(y_j) &= (\pi_i^U)^* \frac{\psi_i'^*(x_j)}{\psi_i'^*(x_r)} = (\pi_i^U)^* \frac{\psi_i^*(x_j) - b(i,j)}{\psi_i^*(x_r) - b(i,r)} \\
&= \frac{\psi_{i+1}^*(x_j) - \psi_{i+1}^*(x_r) b(i+1,j)}{\psi_{i+1}^*(x_r) - \psi_{i+1}^*(x_r) b(i+1,r)} \\
&= \frac{\psi_{i+1}^*(y_j) - b(i+1,j)}{1 - b(i+1,r)}.
\end{aligned}
$$

This implies that

$$
\psi_{i+1}^*(y_j) - \psi_{i+1}'^*(y_j) = \frac{b(i+1,j) - b(i+1,r)\psi_{i+1}^*(y_j)}{1 - b(i+1,r)}
$$

is in $(\Pi_{i+1}^U)_*^{-1}\big(MC(I_0^U), 1\big)$, as required. $\square$

11. TUNING OF IDEALS

Following (14.9) and (77), we are looking for ideals that contain information about all derivatives of I with equalized markings.

Definition 98 (Maximal coefficient ideals). Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. The *maximal coefficient ideal* of order s of I is

$$
W_s(I) := \left(\prod_{j=0}^m \big(D^j(I)\big)^{c_j} : \sum (m-j)c_j \geq s \right) \subset \mathcal{O}_X.
$$

The ideals $W_s(I)$ satisfy a series of useful properties.

Proposition 99. *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \text{max-ord}\, I$. Then*

(1) $W_{s+1}(I) \subset W_s(I)$ *for every s,*
(2) $W_s(I) \cdot W_t(I) \subset W_{s+t}(I)$,
(3) $D(W_{s+1}(I)) = W_s(I)$,
(4) $MC(W_s(I)) = W_1(I) = MC(I)$,
(5) $W_s(I)$ *is MC-invariant,*
(6) $W_s(I) \cdot W_t(I) = W_{s+t}(I)$ *whenever $t \geq (m-1) \cdot \text{lcm}(2, \ldots, m)$ and $s = r \cdot \text{lcm}(2, \ldots, m)$,*
(7) $\big(W_s(I)\big)^j = W_{js}(I)$ *whenever $s = r \cdot \text{lcm}(2, \ldots, m)$ for some $r \geq m-1$, and*
(8) $W_s(I)$ *is D-balanced whenever $s = r \cdot \text{lcm}(2, \ldots, m)$ for some $r \geq m-1$.*

Proof. Assertions (1) and (2) are clear and the inclusion $D(W_{s+1}(I)) \subset W_s(I)$ follows from the product rule. Pick $x_1 \in MC(I)$ that has order 1 at p. Then $x_1^{s+1} \in W_{s+1}(I)$, implying

$$
x_1^s = (s+1)^{-1} \tfrac{\partial}{\partial x_1} x_1^{s+1} \in D\big(W_{s+1}(I)\big).
$$

Next we prove by induction on t that $x_1^{s-t} W_t \subset D\big(W_{s+1}(I)\big)$, which gives (3) for $t = s$.

Note that $x_1^{s+1-t} f \in W_{s+1}(I)$ for any $f \in W_t(I)$. Thus

$$
(s+1-t)x_1^{s-t} f + x_1^{s+1-t}\big(\tfrac{\partial}{\partial x_1} f\big) = \tfrac{\partial}{\partial x_1}\big(x_1^{s+1-t} f\big) \in D\big(W_{s+1}(I)\big).
$$

Since $\frac{\partial}{\partial x_1} f \in W_{t-1}(I)$, then by induction, $x_1^{s+1-t}\left(\frac{\partial}{\partial x_1} f\right) \in D\big(W_{s+1}(I)\big)$. Hence also $x_1^{s-t} f \in D\big(W_{s+1}(I)\big)$.

Applying (3) repeatedly gives that $MC(W_s(I)) = W_1(I)$, which in turn contains $D^{m-1}(I) = MC(I)$ by definition. Conversely, $W_1(I)$ is generated by products of derivatives, at least one of which is a derivative of order $< m$. Thus

$$W_1(I) \subset \sum_{j<m} D^j(I) = D^{m-1}(I),$$

proving (4). Together with (2) and (3), this implies (5).

Thinking of elements of $D^{m-j}(I)$ as variables of degree j, (6) is implied by (99.9), and (7) is a special case of (6).

Finally, if $s = r \cdot \mathrm{lcm}(2, \ldots, m)$ for some $r \geq m - 1$, then using (3) and (7) we get that

$$\left(D^i\big(W_s(I)\big)\right)^s = \big(W_{s-i}(I)\big)^s \subset W_{s(s-i)}(I) = \big(W_s(I)\big)^{s-i}. \quad \square$$

Claim 99.9. Let $u_1, \ldots, u_m$ be variables such that $\deg(u_i) = i$. Then any monomial $U = \prod u_i^{c_i}$ with $\deg(U) \geq (r + m - 1) \cdot \mathrm{lcm}(2, \ldots, m)$ can be written as $U = U_1 \cdot U_2$, where $\deg(U_1) = r \cdot \mathrm{lcm}(2, \ldots, m)$.

Proof. Set $V_i = u_i^{\mathrm{lcm}(2,\ldots,m)/i}$, and write $u_i^{c_i} = V_i^{b_i} \cdot W_i$ for some b_i such that $\deg W_i < \mathrm{lcm}(2, \ldots, m)$.

If $\sum b_i \geq r$, then choose $0 \leq d_i \leq b_i$ such that $\sum d_i = r$, and take $U_1 = \prod V_i^{b_i}$. Otherwise, $\deg U < (r-1) \cdot \mathrm{lcm}(2, \ldots, m) + m \cdot \mathrm{lcm}(2, \ldots, m)$, a contradiction. $\quad \square$

Aside 99.10. Note that one can think of (99.9) as a statement about certain multiplication maps

$$H^0(X, \mathcal{O}_X(a)) \times H^0(X, \mathcal{O}_X(b)) \to H^0(X, \mathcal{O}_X(a+b)),$$

where X is the weighted projective space $\mathbb{P}(1, 2, \ldots, m)$. The above claim is a combinatorial version of the Castelnuovo-Mumford regularity theorem in this case (cf. [Laz04, Sec.1.8]).

It seems to me that (99.6) should hold for $t \geq \mathrm{lcm}(2, \ldots, m)$ and even for many smaller values of t as well.

It is easy to see that $(m-1) \cdot \mathrm{lcm}(2, \ldots, m) \leq m!$ for $m \geq 6$, and one can check by hand that (99.6) holds for $t \geq m!$ for $m = 1, 2, 3, 4, 5$. Thus we conclude that $W_{m!}(I)$ is D-balanced. This is not important, but the traditional choice of the coefficient ideal corresponds to $W_{m!}(I)$.

The following close analog of (77) leads to ideal sheaves that behave the "same" as a given ideal I, as far as order reduction is concerned.

Theorem 100 (Tuning of ideals, I)**.** *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf and $m = \mathrm{max\text{-}ord}\, I$. Let $s \geq 1$ be an integer and J any ideal sheaf satisfying*

$$I^s \subset J \subset W_{ms}(I).$$

Then a smooth blow-up sequence

$$X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

is a smooth blow-up sequence of order $\geq m$ starting with (X, I, m) iff it is a smooth blow-up sequence of order $\geq ms$ starting with (X, J, ms).

Proof. Assume that we get a smooth blow-up sequence starting with (X, I, m):

$$(X_r, I_r, m) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, m) \xrightarrow{\pi_0} (X_0, I_0, m) = (X, I, m).$$

We prove by induction on r that we also get a smooth blow-up sequence starting with (X, J, ms):

$$(X_r, J_r, ms) \xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, ms) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, J_1, ms) \xrightarrow{\pi_0} (X_0, J_0, ms) = (X, J, ms).$$

Assume that this holds up to step $r-1$. We need to show that the last blow-up $\pi_{r-1} : X_r \to X_{r-1}$ is a blow-up for (X_{r-1}, J_{r-1}, ms). That is, we need to show that

$$\operatorname{ord}_Z I_{r-1} \geq m \;\Rightarrow\; \operatorname{ord}_Z J_{r-1} \geq ms \quad \text{for any } Z \subset X_{r-1}.$$

Let $\Pi_{r-1} : X_{r-1} \to X_0$ denote the composite. Since $J \subset W_{ms}(I)$, we know that

$$
\begin{aligned}
J_{r-1} &= \big(\Pi_{r-1}\big)_*^{-1}(J, ms) \\
&\subset \big(\Pi_{r-1}\big)_*^{-1}\big(W_{ms}(I), ms\big) \\
&= \big(\Pi_{r-1}\big)_*^{-1}\Big(\textstyle\prod_j \big(D^j I, m-j\big)^{c_j} : \sum(m-j)c_j \geq ms\Big) \\
&= \Big(\textstyle\prod_j \big((\Pi_{r-1})_*^{-1}(D^j I, m-j)\big)^{c_j} : \sum(m-j)c_j \geq ms\Big) \\
&\subset \Big(\textstyle\prod_j \big(D^j (\Pi_{r-1})_*^{-1}(I, m)\big)^{c_j} : \sum(m-j)c_j \geq ms\Big) \quad \text{by (76)} \\
&= \Big(\textstyle\prod_j \big(D^j (I_{r-1}, m)\big)^{c_j} : \sum(m-j)c_j \geq ms\Big).
\end{aligned}
$$

If $\operatorname{ord}_Z I_{r-1} \geq m$, then $\operatorname{ord}_Z D^j(I_{r-1}) \geq m-j$, and so

$$\operatorname{ord}_Z \prod_j \big(D^j(I_{r-1}, m)\big)^{c_j} \geq \sum(m-j)c_j \geq ms,$$

proving one direction.

In order to prove the converse, let

$$(X_r, J_r, ms) \xrightarrow{\pi_{r-1}} (X_{r-1}, J_{r-1}, ms) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, J_1, ms) \xrightarrow{\pi_0} (X_0, J_0, ms) = (X, J, ms)$$

be a smooth blow-up sequence starting with (X, J, ms). Again by induction we show that it gives a smooth blow-up sequence starting with (X, I, m). Since $I^s \subset J$, we know that

$$I_{r-1}^s = \big((\Pi_{r-1})_*^{-1} I\big)^s \subset \big(\Pi_{r-1}\big)_*^{-1}(J, ms) = J_{r-1}.$$

Thus if $\operatorname{ord}_Z J_{r-1} \geq ms$, then $\operatorname{ord}_Z I_{r-1} \geq m$, and so $\pi_{r-1} : X_r \to X_{r-1}$ is also a blow-up for (X_{r-1}, I_{r-1}, m). $\qquad\square$

Corollary 101 (Tuning of ideals, II)**.** *Let X be a smooth variety, $I \subset \mathcal{O}_X$ an ideal sheaf with $m = \operatorname{max-ord} I$ and E a divisor with simple normal crossings. Let $s = r \cdot \operatorname{lcm}(2, \ldots, m)$ for some $r \geq m-1$. Then $W_s(I)$ is MC-invariant, D-balanced, and a smooth blow-up sequence*

$$X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X$$

is a blow-up sequence of order $\geq m$ starting with (X, I, m, E) iff it is a blow-up sequence of order $\geq s$ starting with $(X, W_s(I), s, E)$.

Proof. Everything follows from (99) and (100), except for the role played by E.

Adding E to (X, I) (resp., to $(X, W_s(I))$) means that now we can use only blow-ups whose centers are in simple normal crossing with E and its total transforms. This poses the same restriction on smooth blow-up sequences for (X, I, E) as on smooth blow-up sequences for $(X, W_s(I), E)$. $\square$

12. Order reduction for ideals

In this section we prove the first main implication (70.1) of the inductive proof. We start with a much weaker result. Instead of getting rid of all points of order m, we prove only that the set of points of order m moves away from the birational transform of a given divisor E^j.

Lemma 102. *Assume that (69) holds in dimensions $< n$. Then for every m, j there is a smooth blow-up sequence functor $\mathcal{BD}_{n,m,j}$ of order m that is defined on triples (X, I, E) with $\dim X = n$, $\max\text{-ord}\, I \leq m$ and $E = \sum_i E^i$ such that if $\mathcal{BD}_{n,m,j}(X, I, E) =$*

$$\Pi : (X_r, I_r, E_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),$$

then

    (1) $\text{cosupp}(I_r, m) \cap \Pi_*^{-1} E^j = \emptyset$, *and*

    (2) $\mathcal{BD}_{n,m,j}$ *commutes with smooth morphisms (34.1) and also with change of fields (34.2).*

Assume in addition that there is an ideal $J \subset \mathcal{O}_{E^j}$ such that J is nonzero on every irreducible component of E^j and $\tau_*(\mathcal{O}_{E^j}/J) = \mathcal{O}_X/I$, where $\tau : E^j \hookrightarrow X$ is the natural injection. Then

    (3) $\mathcal{BD}_{n,1,j}(X, I, E) := \tau_* \mathcal{BMO}_{n-1,1}(E^j, J, 1, (E - E^j)|_{E^j})$.

Proof. By (101), $W_{m!}(I)$ is D-balanced and order reduction for (X, I, E) is equivalent to order reduction for $(X, W_{m!}(I), E)$. Thus from now on we assume that I is D-balanced.

Let Z_{-1} be the union of those irreducible components $E^{jk} \subset E^j$ that are contained in $\text{cosupp}(I, m)$. Let $\pi_{-1} : X_0 \to X$ be the blow-up of Z_{-1}. The blow-up is an isomorphism, but the order of I along E^{jk} is reduced by m and we get a new ideal sheaf I_0. Since $\max\text{-ord}_{E^{jk}}\, I \leq m$ to start with, $\max\text{-ord}_{E^{jk}}\, I_0 = \max\text{-ord}_{E^{jk}}\, I - m \leq 0$. Thus $\text{cosupp}(I_0, m)$ does not contain any irreducible component of E^j.

Next, set $S := E^j$ with injection $\tau : S \hookrightarrow X$, $E_S := (E - E^j)|_S$ and consider the triple $(S, I_0|_S, E_S)$. By the going-up theorem (84), every blow-up sequence of order $\geq m$ starting with $(S, I_0|_S, m, (E - E^j)|_S)$ corresponds to a blow-up sequence of order m starting with $(X_0, I_0, E - E^j)$. Since $S = E^j$, every blow-up center is a smooth subvariety of the birational transform of E^j; thus we in fact get a blow-up sequence of order m starting with (X_0, I_0, E). Set

$$\mathcal{BD}_{n,m,j}(X, I, E) := \tau_* \mathcal{BMO}_{n-1,m}(S, I_0|_S, m, E_S) \xrightarrow{\pi_{-1}} X.$$

That is, we take $\mathcal{BMO}_{n-1,m}(S, I_0|_S, m, (E - E^j)|_S)$, push it forward (30.3) and compose the resulting blow-up sequence on the right with our first blow-up π_{-1}. (This is the reason for the subscript -1.) By (85) we obtain

$$\Pi_r : X_r \to X \quad \text{with} \quad I_r := (\Pi_r)_*^{-1} I, \; E_r := (\Pi_r)_{\text{tot}}^{-1}(E)$$

such that $(\Pi_r)_*^{-1}(E^j)$ is disjoint from $\mathrm{cosupp}(I_r, m)$.

The functoriality properties of $\mathcal{BD}_{n,m,j}(X, I, E)$ follow from the corresponding functoriality properties of $\mathcal{BMO}_{n-1,m}(S, I_0|_S, E_S)$. All the steps are obvious, but for the first time, let us go through the details.

Let $h : Y \to X$ be a smooth surjection. Set $E_Y^j := h^{-1}(E^j)$. Then $h|_{E_Y^j} : E_Y^j \to E^j$ is also a smooth surjection and we get the same result whether we first pull back by h and then restrict to E_Y^j or we first restrict to E^j and then pull back by $h|_{E_Y^j}$. That is,

$$\left(h|_{E_Y^j}\right)^*\left(E^j, I_0|_{E^j}, m, (E - E^j)|_{E^j}\right) = \left(E_Y^j, (h^*I)_0|_{E_Y^j}, h^{-1}(E - E^j)|_{E_Y^j}\right).$$

Therefore,

$$\begin{aligned}
\mathcal{BMO}_{n-1,m}\big(E_Y^j, (h^*I)_0|_{E_Y^j}, h^{-1}(E - E^j)|_{E_Y^j}\big) \\
= \big(h|_{E_Y^j}\big)^* \mathcal{BMO}_{n-1,m}\big(E^j, I_0|_{E^j}, m, (E - E^j)|_{E^j}\big),
\end{aligned}$$

and hence

$$h^* \mathcal{BD}_{n,m,j}(X, I, E) = \mathcal{BD}_{n,m,j}(Y, h^*I, h^{-1}(E)).$$

If $h : Y \to X$ is any smooth morphism, we see similarly that the same blow-ups end up with empty centers.

The functoriality property (34.2) holds since change of the base field commutes with restrictions.

Assume finally that $\mathcal{O}_X/I = \tau_*(\mathcal{O}_{E^j}/J)$. Every local equation of E^j is an order 1 element in I. Thus $m = \mathrm{max\text{-}ord}\, I = 1$ and so $W_{m!}(I) = I$. If J is nonzero on every irreducible component of E^j then $Z_{-1} = \emptyset$ and so $I_0 = I$, proving (3). $\qquad\square$

The main theorem of this section is the following.

Theorem 103. *Assume that (69) holds in dimensions $< n$. Then for every m there is a smooth blow-up sequence functor $\mathcal{BO}_{n,m}$ of order m that is defined on triples (X, I, E) with $\dim X = n$ and $\mathrm{max\text{-}ord}\, I \leq m$ such that if $\mathcal{BO}_{n,m}(X, I, E) =$*

$$\begin{aligned}
\Pi : (X_r, I_r, E_r) &\xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \\
&\xrightarrow{\pi_1} (X_1, I_1, E_1) \xrightarrow{\pi_0} (X_0, I_0, E_0) = (X, I, E),
\end{aligned}$$

then

(1) $\mathrm{max\text{-}ord}\, I_r < m$, *and*
(2) $\mathcal{BO}_{n,m}$ *commutes with smooth morphisms (34.1) and also with change of fields (34.2).*

Assume in addition that there is a smooth hypersurface $\tau : Y \hookrightarrow X$ and an ideal sheaf $J \subset \mathcal{O}_Y$ such that J is nonzero on every irreducible component of Y and $\tau_*(\mathcal{O}_Y/J) = \mathcal{O}_X/I$. Then $\mathrm{max\text{-}ord}\, I = 1$ and

(3) $\mathcal{BO}_{n,1}(X, I, \emptyset) = \tau_* \mathcal{BMO}_{n-1,1}(Y, J, 1, \emptyset).$

The proof is done in three steps.

Step 1 (Tuning I). By (101), there is an ideal $W(I) = W_s(I)$ for suitable s, which is D-balanced, MC-invariant and order reduction for (X, I, E) is equivalent to order reduction for $(X, W(I), E)$. (Let us take $s = m!$ to avoid further choices.) Thus from now on we assume that I is D-balanced and MC-invariant.

Step 2 (Maximal contact case). Here we assume that there is a smooth hypersurface of maximal contact $H \subset X$. This is always satisfied in a suitable open

neighborhood of any point by (80.2), but it may hold globally as well. This condition is also preserved under disjoint unions.

Under a smooth blow-up of order m, the birational transform of a smooth hypersurface of maximal contact is again a smooth hypersurface of maximal contact; thus we stay in the maximal contact case.

We intend to restrict everything to H, but we run into the problem that $E|_H$ need not be a simple normal crossing divisor. We take care of this problem first.

Step 2.1. If $E = \sum_{i=1}^{s} E^i$, we apply (102) to each E^i. At the end we get a blow-up sequence $\Pi : X_r \to X$ such that $\mathrm{cosupp}(I_r, m)$ is disjoint from $\Pi_*^{-1}E$.

Note that the new exceptional divisors obtained in the process (and added to E) have simple normal crossings with the birational transforms of H, so $H_r + E_r$ is a simple normal crossing divisor. (I have used the ordering of the index set of E. This is avoided traditionally by restricting (X, I, E) successively to the multiplicity $n - j$ locus of E, starting with the case $j = 0$. The use of the ordering cannot be avoided in (111.3), so there is not much reason to go around it here.)

Step 2.2. Once $H + E$ is a simple normal crossing divisor, we restrict everything to the birational transform of H, and we obtain order reduction using dimension induction and (102).

Step 3 (Global case). There may not be a global smooth hypersurface of maximal contact $H \subset X$, but we can cover X with open subsets $X^{(j)} \subset X$ such that on each $X^{(j)}$ there is a smooth hypersurface of maximal contact $H^{(j)} \subset X^{(j)}$. Thus the disjoint union

$$H^* := \coprod_j H^{(j)} \subset \coprod_j X^{(j)} =: X^*$$

is a smooth hypersurface of maximal contact. Let $g : X^* \to X$ be the coproduct of the injections $X^{(j)} \hookrightarrow X$.

By the previous step $\mathcal{BO}_{n,m}$ is defined on $(X^*, g^*I, g^{-1}E)$. Then we argue as in (37) to prove that $\mathcal{BO}_{n,m}(X^*, g^*I, g^{-1}E)$ descends to give $\mathcal{BO}_{n,m}(X, I, E)$.

Only Steps 2 and 3 need amplification.

104 (Step 2, Maximal contact case). We start with a triple (X, I, E), where I is D-balanced and MC-invariant, and assume that there is a smooth hypersurface of maximal contact $H \subset X$. Set $m = \mathrm{max\text{-}ord}\, I$.

Warning. As we blow up, we get birational transforms of I which may be neither D-balanced nor MC-invariant. We do not attempt to "fix" this problem, since the relevant consequences of these properties (84) and (92) are established for any sequence of blow-ups of order m. This also means that we should not pick new hypersurfaces of maximal contact after a blow-up but rather stick with the birational transforms of the old ones.

Step 2.1 (Making $\mathrm{cosupp}(I_r, m)$ and $\Pi_*^{-1}E$ disjoint). To fix notation, write $E = \sum_{i=1}^{s} E^i$, and set $(X_0, I_0, E_0) := (X, I, E)$ and $H_0 := H$. The triple (X_0, I_0, E_0) satisfies the assumptions of Step 2.1.1.

Step 2.1.j. Assume that we have already constructed a smooth blow-up sequence of order m starting with (X_0, I_0, E_0) whose end result is

$$\begin{aligned}
&\Pi_{r(j-1)} : X_{r(j-1)} \to \cdots \to X_0, \quad \text{where} \\
&I_{r(j-1)} := \left(\Pi_{r(j-1)}\right)_*^{-1} I \quad \text{and} \quad E_{r(j-1)} := \left(\Pi_{r(j-1)}\right)_{\mathrm{tot}}^{-1}(E),
\end{aligned}$$

such that

$$\left(\Pi_{r(j-1)}\right)_*^{-1}(E^i) \cap \mathrm{cosupp}(I_{r(j-1)}, m) = \emptyset \quad \text{for } i < j.$$

Apply (102) to $\big(X_{r(j-1)}, I_{r(j-1)}, E_{r(j-1)}\big)$ and the divisor $E^j_{r(j-1)}$ to obtain

$$\Pi_{r(j)} : X_{r(j)} \to X_{r(j-1)} \cdots \to X_0$$

such that

$$\big(\Pi_{r(j)}\big)^{-1}_* (E^i) \cap \operatorname{cosupp}(I_{r(j)}, m) = \emptyset \quad \text{for } i \le j.$$

Note that the center of every blow-up is contained in every hypersurface of maximal contact. Thus $H_{r(j)} := \big(\Pi_{r(j)}\big)^{-1}_* H$ is a smooth hypersurface of maximal contact, and every new divisor in $\big(\Pi_{r(j)}\big)^{-1}_{\text{tot}} E$ is transversal to $H_{r(j)}$. If $E = \sum_{i=1}^s E^i$, then after Step 2.1.s, we have achieved that

- $\big(\Pi_{r(s)}\big)^{-1}_* E$ is disjoint from $\operatorname{cosupp}(I_{r(s)}, m)$, and
- for any hypersurface of maximal contact $H \subset X$, the divisor $H_{r(s)} + E_{r(s)}$ has simple normal crossing along $\operatorname{cosupp}(I_{r(s)}, m)$.

Note that we perform all these steps even if $H + E$ is a simple normal crossing divisor to start with, though in this case they do not seem to be necessary. We would, however, run into problems with the compatibility of the numbering in the blow-up sequences otherwise.

Step 2.2 (Restricting to H). After dropping the subscript $r(s)$ we have a triple (X, I, E) and a smooth hypersurface of maximal contact $H \subset X$ such that $H + E$ is also a simple normal crossing divisor. We can again replace I by $W(I)$ and thus assume that I is MC-invariant. Note that we do not pick a new hypersurface of maximal contact, but use only the birational transforms $H_{r(s)}$ of the old hypersurfaces of maximal contact.

Declare $E^0 := H$ to be the first divisor in $H + E$ and apply (102) to $(X, I, H + E)$ with $j = 0$. This gives a sequence of blow-ups $\Pi : X_r \to X$ such that $\operatorname{cosupp} \Pi^{-1}_*(I, m)$ is disjoint from $\Pi^{-1}_* H$. However, H is a smooth hypersurface of maximal contact, and hence, by definition, $\operatorname{cosupp} \Pi^{-1}_*(I, m) \subset \Pi^{-1}_* H$. Thus $\operatorname{cosupp} \Pi^{-1}_*(I, m) = \emptyset$, as we wanted.

Step 2.3 (Functoriality). Assuming functoriality in dimension $< n$, we have functoriality in Step 2.1 by the corresponding functoriality in (102).

In Step 2.2 we rely on the choice of a hypersurface of maximal contact H, which is not unique. Let H, H' be two hypersurfaces of maximal contact such that $H + E$ and $H' + E$ are both simple normal crossing divisors. We can use either of the two blow-up sequences $\mathcal{BD}_{n,m,0}(X, I, H + E)$ and $\mathcal{BD}_{n,m,0}(X, I, H' + E)$ to construct $\mathcal{BO}_{n,m}(X, I, E)$.

Here we need that I is MC-invariant. By (92) this implies that $(X, I, H + E)$ and $(X, I, H' + E)$ are étale equivalent. Thus the two blow-up sequences $\mathcal{BD}_{n,m,0}(X, I, H + E)$ and $\mathcal{BD}_{n,m,0}(X, I, H' + E)$ are also étale equivalent. By (97) this implies that these blow-up sequences are identical.

As we noted in (34), the functoriality package is local, so we do not have to consider it separately in the next step.

Step 2.4 (Closed embeddings) Let $\tau : Y \hookrightarrow X$ be a smooth hypersurface and $J \subset \mathcal{O}_Y$ an ideal sheaf such that J is nonzero on every irreducible component of Y and $\tau_*(\mathcal{O}_Y/J) = \mathcal{O}_X/I$. Then I contains the local equations of Y, and so it has order 1. In particular, $I = W(I)$. If $E = \emptyset$ then Step 2.1 does nothing, and in Step 2.2 we can choose $H = Y$. Thus (103.3) follows from (102.3).

As in (37), going from the local to the global case is essentially automatic. For ease of reference, let us axiomatize the process.

Theorem 105 (Globalization of blow-up sequences). *Assume that we have the following:*

(1) *a class of smooth morphisms $\mathcal{M}$ that is closed under fiber products and coproducts (for instance, $\mathcal{M}$ could be all smooth morphisms, all étale morphisms or all open immersions);*

(2) *two classes of triples $\mathcal{GT}$ (global triples) and $\mathcal{LT}$ (local triples) such that*

(i) *for every $(X, I, E) \in \mathcal{GT}$ and every $x \in X$ there is an $\mathcal{M}$-morphism $g_x : (x' \in U_x) \to (x \in X)$ such that $(U_x, g^*I, g^{-1}E)$ is in $\mathcal{LT}$, and*

(ii) *$\mathcal{LT}$ is closed under disjoint unions;*

(3) *a blow-up sequence functor $\mathcal{B}$ defined on $\mathcal{LT}$ that commutes with surjections in $\mathcal{M}$.*

Then $\mathcal{B}$ has a unique extension to a blow-up sequence functor $\overline{\mathcal{B}}$, which is defined on $\mathcal{GT}$ and which commutes with surjections in $\mathcal{M}$.

Proof. For any $(X, I, E) \in \mathcal{GT}$ choose $\mathcal{M}$-morphisms $g_{x_i} : U_{x_i} \to X$ such that the images cover X.

Let $X' := \coprod_i U_{x_i}$ be the disjoint union and $g : X' \to X$ the induced $\mathcal{M}$-morphism. By assumption $(X', g^*I, g^{-1}E) \in \mathcal{LT}$.

Set $X'' := X' \times_X X'$. By assumption the two coordinate projections $\tau_1, \tau_2 : X'' \to X'$ are in $\mathcal{M}$ and are surjective.

The blow-up sequence $\mathcal{B}$ for X' starts with blowing up $Z'_0 \subset X'$, and the blow-up sequence $\mathcal{B}$ for X'' starts with blowing up $Z''_0 \subset X''$. Since $\mathcal{B}$ commutes with the τ_i, we conclude that

$$\tau_1^*(Z'_0) = Z''_0 = \tau_2^*(Z'_0). \tag{105.4}$$

If $\mathcal{M} = \{\text{open immersions}\}$, we have proved in (37) that the subschemes $Z'_0 \cap U_{x_i} \subset X$ glue together to a subscheme $Z_0 \subset X$. This is the only case we need for the proof of (103).

The conclusion still holds for any $\mathcal{M}$, but we have to use the theory of faithfully flat descent; see [Gro95] or [Mur67, Ch.VII].

This way we obtain $X_1 := B_{Z_0} X$ such that $X'_1 = X' \times_X X_1$. We can repeat the above argument to obtain the center $Z_1 \subset X_1$ and eventually get the whole blow-up sequence for (X, I, E). □

The following example, communicated to me by Bierstone and Milman, shows that while principalization proceeds by smooth blow-ups, the resolution of singularities also involves blowing up singular centers.

Example 106. Consider the subvariety $X \subset \mathbb{A}^4$ defined by the ideal $I = (x^3 - y^2, x^4 + xz^2 - w^3)$. Let us see how the principalization proceeds.

Note that $\text{ord}\, I = 2$ and $H = (y = 0)$ is a hypersurface of maximal contact. $I|_H = (x^3, xz^2 - w^3)$ has order 3 and $MC(I|_H) = (x, z, w)$. Thus the first step is to blow up the origin in $\mathbb{A}^4$.

Consider the chart $x_1 = x, y_1 = y/x, z_1 = z/x, w_1 = w/x$. The birational transform of I is $I_1 = (x_1 - y_1^2, x_1(x_1 + z_1^2 - w_1^3))$ and $E_1 = (x_1 = 0)$. The order has dropped to 1, so we continue with $(I_1, 1, E_1)$.

Since $\text{cosupp}(I_1, 1)$ is not disjoint from E_1, we proceed as in (104.1). The restriction is $I_1|_{E_1} = (x_1, y_1^2)$, and thus next we have to blow up $(x_1 = y_1 = 0)$.

On the other hand, the birational transform of X is

$$X_1 = (x_1 - y_1^2 = x_1 + z_1^2 - w_1^3 = 0).$$

Its intersection with $(x_1 = y_1 = 0)$ is the cuspidal curve $(x_1 = y_1 = z_1^2 - w_1^3 = 0)$. Thus the resolution of X first blows up the origin and then the new exceptional curve, which is singular.

13. ORDER REDUCTION FOR MARKED IDEALS

In this section we prove the second main implication (70.2) of the inductive proof. That is, we prove the following.

Theorem 107. *Assume that (68) holds in dimensions $\leq n$. Then for every m, there is a smooth blow-up sequence functor $\mathcal{BMO}_{n,m}$ defined on triples (X, I, m, E) with $\dim X = n$ such that if $\mathcal{BMO}_{n,m}(X, I, m, E) =$*

$$\Pi : (X_r, I_r, m, E_r) \xrightarrow{\pi_{r-1}} (X_{r-1}, I_{r-1}, m, E_{r-1}) \xrightarrow{\pi_{r-2}} \cdots$$
$$\xrightarrow{\pi_1} (X_1, I_1, m, E_1) \xrightarrow{\pi_0} (X_0, I_0, m, E_0) = (X, I, m, E),$$

then

(1) $\text{max-ord}\, I_r < m$,
(2) $\mathcal{BMO}_{n,m}$ *commutes with smooth morphisms (34.1) and with change of fields (34.2), and*
(3) *if $m = \text{max-ord}\, I$ then $\mathcal{BMO}_{n,m}(X, I, m, \emptyset) = \mathcal{BO}_{n,m}(X, I, \emptyset)$.*

Before proving (107), we show that it implies the two claims in (71).

108 (Proof of (71.1–2)). First, (71.1) is the same as (107.3).

The claimed identity in (71.2) is a local question on X, thus we may assume that there is a chain of smooth subvarieties $Y = Y_0 \subset Y_1 \subset \cdots \subset Y_c = X$ such that each is a hypersurface in the next one. Thus it is enough to prove the case when Y is a hypersurface in X.

Every local equation of Y is in I, thus $\text{max-ord}\, I = 1$. Therefore, $\mathcal{BMO}_{\dim X, 1}(X, I, 1, \emptyset) = \mathcal{BO}_{\dim X, 1}(X, I, \emptyset)$ by (107.3) and (103.3) gives that $\mathcal{BO}_{\dim X, 1}(X, I, \emptyset) = \tau_* \mathcal{BMO}_{\dim Y, 1}(Y, J, 1, \emptyset)$. Putting the two together gives (71.2). $\square$

109 (Plan of the proof of (107)). *Step 1.* We start with the unmarked triple (X, I, E), and using (68) in dimension n, we reduce its order below m. That is, we get a composite of smooth blow-ups $\Pi_1 : X^1 \to X$ such that $(\Pi_1)_*^{-1} I$ has order $< m$. The problem is that $(\Pi_1)_*^{-1} I$ differs from $(\Pi_1)_*^{-1}(I, m)$ along the exceptional divisors of Π_1, and the latter can have very high order. We decide not to worry about it for now.

Step 2. Continuing with $(X^1, (\Pi_1)_*^{-1}(I, m), (\Pi_1)_{\text{tot}}^{-1} E)$, we blow up subvarieties where the birational transform of (I, m) has order $\geq m$, and the birational transform of I has order ≥ 1.

Eventually we get $\Pi_2 : X^2 \to X$ such that $\text{cosupp}(\Pi_2)_*^{-1} I$ is disjoint from the locus where $(\Pi_2)_*^{-1}(I, m)$ has order $\geq m$. We can now completely ignore $(\Pi_2)_*^{-1} I$. Since $(\Pi_2)_*^{-1} I$ and $(\Pi_2)_*^{-1}(I, m)$ agree up to tensoring with the ideal sheaf of a divisor whose support is in E_r, we can assume from now on that $(\Pi_2)_*^{-1}(I, m)$ is the ideal sheaf of a divisor with simple normal crossing.

Step 3. Order reduction for the marked ideal sheaf of a divisor with simple normal crossing is rather easy.

Instead of strictly following this plan, we divide the ideal into a "simple normal crossing part" and the "rest" using all of E, instead of exceptional divisors only. This is solely a notational convenience.

Definition–Lemma 110. Given (X, I, E), we can write I uniquely as $I = M(I) \cdot N(I)$, where $M(I) = \mathcal{O}_X(-\sum c_i E^i)$ for some c_i and cosupp $N(I)$ does not contain any of the E^i. $M(I)$ is called the *monomial part* of I and $N(I)$ the *nonmonomial part* of I.

Since the E^i are not assumed irreducible, cosupp $N(I)$ may contain irreducible components of some of the E^i.

111 (Proof of (107)). We write $I = M(I) \cdot N(I)$ and try to deal with the two parts separately.

Step 1 (Reduction to ord $N(I) < m$). If ord $N(I) \geq m$, we can apply order reduction (68) to $N(I)$, until its order drops below m. This happens at some $\Pi_1 : X^1 \to X$. Note that the two birational transforms

$$(\Pi_1)_*^{-1} N(I) \quad \text{and} \quad (\Pi_1)_*^{-1}(I, m)$$

differ only by tensoring with an ideal sheaf of exceptional divisors of Π_1, thus only in their monomial part. Therefore,

$$N\big((\Pi_1)_*^{-1}(I, m)\big) = (\Pi_1)_*^{-1} N(I),$$

and so we have reduced to the case where the maximal order of the nonmonomial part is $< m$.

To simplify notation, instead of $(X^1, (\Pi_1)_*^{-1}(I, m), (\Pi_1)_{\text{tot}}^{-1}(E))$, write (X, I, m, E). From now on we may assume that max-ord $N(I) < m$.

Step 2 (Reduction to $\text{cosupp}(I, m) \cap \text{cosupp } N(I) = \emptyset$). Our aim is to continue with order reduction further and get rid of $N(I)$ completely. The problem is that we are allowed to blow up only subvarieties along which (I, m) has order at least m. Thus we can blow up $Z \subset X$ with $\text{ord}_Z N(I) < m$ only if $\text{ord}_Z I \geq m$. We will be able to guarantee this interplay by a simple trick.

Let s be the maximum order of $N(I)$ along $\text{cosupp}(I, m)$. We reduce this order step-by-step, eventually ending up with $s = 0$, which is the same as $\text{cosupp}(I, m) \cap \text{cosupp } N(I) = \emptyset$.

It would not have been difficult to develop order reduction theory for several marked ideals and to apply it to the pair of marked ideals $(N(I), s)$ and (I, m), but the following simple observation reduces the general case to a single ideal:

$$\text{ord}_Z J_1 \geq s \text{ and } \text{ord}_Z J_2 \geq m \Leftrightarrow \text{ord}_Z(J_1^m + J_2^s) \geq ms.$$

Thus we apply order reduction to the ideal $N(I)^m + I^s$, which has order $\geq ms$. Every smooth blow-up sequence of order ms starting with $N(I)^m + I^s$ is also a smooth blow-up sequence of order s starting with $N(I)$ and a smooth blow-up sequence of order m starting with I. Thus we stop after $r = r(m, s)$ steps when we have achieved $\text{cosupp}(I_r, m) \cap \text{cosupp}(N(I_r), s) = \emptyset$. We can continue with $s - 1$ and so on.

Eventually we achieve a situation where (after dropping the subscript) the cosupports of $N(I)$ and of (I, m) are disjoint. Since the center of any further blow-up is contained in $\text{cosupp}(I, m)$, we can replace X by $X \setminus \text{cosupp } N(I)$ and thus assume that $I = M(I)$. The final step is now to deal with monomial ideals.

Step 3 (Order reduction for $M(I)$). Let X be a smooth variety, $\cup_{j \in J} E^j$ a simple normal crossing divisor with ordered index set J and a_j natural numbers giving the monomial ideal $I := \mathcal{O}_X(-\sum a_j E^j)$.

The usual method would be to look for the highest multiplicity locus and blow it up. This, however, does not work, not even for surfaces; see (112).

The only thing that saves us at this point is that the divisors E^i come with an ordered index set. This allows us to specify in which order to blow up. There are many possible choices. As far as I can tell, there is no natural or best variant.

Step 3.1. Find the smallest j such that $a_j \geq m$ is maximal. If there is no such j, go to the next step. Otherwise, blow up E^j. Repeating this, we eventually get to the point where $a_j < m$ for every j.

Step 3.2. Find the lexicographically smallest $(j_1 < j_2)$ such that $E^{j_1} \cap E^{j_2} \neq \emptyset$ and $a_{j_1} + a_{j_2} \geq m$ is maximal. If there is no such $(j_1 < j_2)$, go to the next step. Otherwise, blow up $E^{j_1} \cap E^{j_2}$. We get a new divisor, and put it last as E^{j_ℓ}. Its coefficient is $a_{j_\ell} = a_{j_1} + a_{j_2} - m < m$. The new pairwise intersections are $E^i \cap E^{j_\ell}$ for certain values of i. Note that

$$a_i + a_{j_\ell} = a_i + a_{j_1} + a_{j_2} - m < a_{j_1} + a_{j_2},$$

since $a_i < m$ for every i by Step 3.1.

At each repetition, the pair $(m_2(E), n_2(E))$ decreases lexicographically where

$$
\begin{aligned}
m_2(E) &:= \max\{a_{j_1} + a_{j_2} : E^{j_1} \cap E^{j_2} \neq \emptyset\}, \\
n_2(E) &:= \text{number of } (j_1 < j_2) \text{ achieving the maximum.}
\end{aligned}
$$

Eventually we reach the stage where $a_{j_1} + a_{j_2} < m$ whenever $E^{j_1} \cap E^{j_2} \neq \emptyset$.

Step 3.r. Assume that for every $s < r$ we already have the property

$$a_{j_1} + \cdots + a_{j_s} < m \quad \text{if} \quad j_1 < \cdots < j_s \text{ and } E^{j_1} \cap \cdots \cap E^{j_s} \neq \emptyset. \qquad (*_s)$$

Find the lexicographically smallest $(j_1 < \cdots < j_r)$ such that $E^{j_1} \cap \cdots \cap E^{j_r} \neq \emptyset$ and $a_{j_1} + \cdots + a_{j_r} \geq m$ is maximal. If there is no such $(j_1 < \cdots < j_r)$, go to the next step. Otherwise, blow up $E^{j_1} \cap \cdots \cap E^{j_r}$, and put the new divisor E^{j_ℓ} last with coefficient $a_{j_1} + \cdots + a_{j_r} - m$. As before, the new r-fold intersections are of the form $E^{i_1} \cap \cdots \cap E^{i_{r-1}} \cap E^{j_\ell}$, where $E^{i_1} \cap \cdots \cap E^{i_{r-1}} \neq \emptyset$. Moreover,

$$a_{i_1} + \cdots + a_{i_{r-1}} + a_{j_\ell} = \big(a_{i_1} + \cdots + a_{i_{r-1}} - m\big) + a_{j_1} + \cdots + a_{j_r},$$

which is less than $a_{j_1} + \cdots + a_{j_r}$ since $a_{i_1} + \cdots + a_{i_{r-1}} < m$ by Step $3.r - 1$. Thus the pair $(m_r(E), n_r(E))$ decreases lexicographically, where

$$
\begin{aligned}
m_r(E) &:= \max\{a_{j_1} + \cdots + a_{j_r} : E^{j_1} \cap \cdots \cap E^{j_r} \neq \emptyset\}, \\
n_r(E) &:= \text{number of } (j_1 < \cdots < j_r) \text{ achieving the maximum.}
\end{aligned}
$$

Eventually we reach the stage where the property $(*_r)$ also holds. We can now move to the next step.

At the end of Step 3.n we are done, where $n = \dim X$.

The functoriality conditions are just as obvious as before.

The process greatly simplifies if $m = \text{max-ord}\, I$ and $E = \emptyset$. First, if $E = \emptyset$ then $N(I) = I$. Thus in Step 1 we apply $\mathcal{BO}_{n,m}(X, I, \emptyset)$. Each blow-up has order m, and thus the birational transforms of I agree with the birational transforms of (I, m). At the end of Step 1, $(\Pi_1)_*^{-1} I = (\Pi_1)_*^{-1}(I, m)$. Thus

$$\text{cosupp}\big((\Pi_1)_*^{-1}(I, m)\big) = \emptyset \quad \text{and} \quad M\big((\Pi_1)_*^{-1}(I, m)\big) = \mathcal{O}_{X^1}.$$

Steps 2 and 3 do nothing, and so $\mathcal{BMO}_{n,m}(X, I, m, \emptyset) = \mathcal{BO}_{n,m}(X, I, \emptyset)$. $\qquad \square$

Example 112. Let S be a smooth surface, E^1, E^2 two 2 curves intersecting at a point $p = E^1 \cap E^2$ and $a_1 = a_2 = m + 1$. Let $\pi : S_3 \to S$ be the blow-up of p with exceptional curve E^3. Then

$$\pi_*^{-1}\big(\mathcal{O}_S(-(m+1)(E^1 + E^2)), m\big) = \big(\mathcal{O}_{S_3}(-(m+1)(E^1 + E^2) - (m+2)E^3), m\big).$$

Next we blow up the intersection point $E^2 \cap E^3$ and so on. After $r - 2$ steps we get a birational transform

$$\big(\mathcal{O}_{S_r}(-\textstyle\sum_{i=1}^r (m + p_i)E^i), m\big),$$

where p_i is the ith Fibonacci number. Thus we get higher and higher multiplicity ideals.

References

[Abh66]   Shreeram Shankar Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Pure and Applied Math., Vol. 24, Academic Press, New York, 1966.

[AdJ97]   D. Abramovich and A. J. de Jong, *Smoothness, semistability, and toroidal geometry*, J. Algebraic Geom. **6** (1997), no. 4, 789–801.

[AHV75]   José M. Aroca, Heisuke Hironaka, and José L. Vicente, *The theory of the maximal contact*, Memorias de Matemática del Instituto "Jorge Juan", vol. 29, Instituto "Jorge Juan" de Matemáticas, Consejo Superior de Investigaciones Científicas, Madrid, 1975.

[AHV77]   ———, *Desingularization theorems*, Memorias de Matemática del Instituto "Jorge Juan", vol. 30, Consejo Superior de Investigaciones Científicas, Madrid, 1977.

[AM69]   M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.

[Art69]   M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–58.

[AW97]   Dan Abramovich and Jianhua Wang, *Equivariant resolution of singularities in characteristic* 0, Math. Res. Lett. **4** (1997), no. 2–3, 427–433.

[BM89]   Edward Bierstone and Pierre D. Milman, *Uniformization of analytic spaces*, J. Amer. Math. Soc. **2** (1989), no. 4, 801–836.

[BM91]   ———, *A simple constructive proof of canonical resolution of singularities*, Effective methods in algebraic geometry (Castiglioncello, 1990), Progr. Math., vol. 94, Birkhäuser, Boston, MA, 1991, pp. 11–30.

[BM97]   ———, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[BM03]   ———, *Desingularization algorithms. I. Role of exceptional divisors*, Mosc. Math. J. **3** (2003), no. 3, 751–805, 1197.

[BP96]   Fedor A. Bogomolov and Tony G. Pantev, *Weak Hironaka theorem*, Math. Res. Lett. **3** (1996), no. 3, 299–307.

[BS00a]   Gábor Bodnár and Josef Schicho, *Automated resolution of singularities for hypersurfaces*, J. Symbolic Comput. **30** (2000), no. 4, 401–428.

[BS00b]   ———, *A computer program for the resolution of singularities*, Resolution of singularities (Obergurgl, 1997), Progr. Math., vol. 181, Birkhäuser, Basel, 2000, pp. 231–238.

[BV01]   A. Bravo and O. Villamayor, *Strengthening the theorem of embedded desingularization*, Math. Res. Lett. **8** (2001), no. 1-2, 79–89.

[CGO84]   Vincent Cossart, Jean Giraud, and Ulrich Orbanz, *Resolution of surface singularities*, Lecture Notes in Math., vol. 1101, Springer-Verlag, Berlin, 1984.

[Cut04]   Steven Dale Cutkosky, *Resolution of singularities*, Graduate Studies in Math., vol. 63, American Math. Soc., Providence, RI, 2004.

[dJ96]   A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51–93.

[EH02]   Santiago Encinas and Herwig Hauser, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. **77** (2002), no. 4, 821–845.

[EV98]   Santiago Encinas and Orlando Villamayor, *Good points and constructive resolution of singularities*, Acta Math. **181** (1998), no. 1, 109–158.

[EV03]   ———, *A new proof of desingularization over fields of characteristic zero*, Rev. Mat. Iberoamericana **19** (2003), no. 2, 339–353.

[FKP05]   Anne Frühbis-Krüger and Gerhard Pfister, *Auflösung von Singularitäten*, Mitt. Dtsch. Math.-Ver. **13** (2005), no. 2, 98–105.

[Ful93]   William Fulton, *Introduction to toric varieties*, Annals of Math. Stud., vol. 131, Princeton Univ. Press, Princeton, NJ, 1993.

[Gir74]   Jean Giraud, *Sur la théorie du contact maximal*, Math. Z. **137** (1974), 285–310.

[Gir95]   ———, *Résolution des singularités (d'après Heisuke Hironaka)*, Séminaire Bourbaki, 10, Soc. Math. France, Paris, 1995, pp. Exp. No. 320, 101–113.

[GR71]   H. Grauert and R. Remmert, *Analytische Stellenalgebren*, Springer-Verlag, Berlin, 1971, Unter Mitarbeit von O. Riemenschneider, Die Grundlehren der mathematischen Wissenschaften, 176.

[Gro95]   Alexander Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique, I: Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki, 5, Soc. Math. France, Paris, 1995, pp. Exp. No. 190, 299–327.

[Har77]   Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.

[Hau03]   Herwig Hauser, *The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand)*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), no. 3, 323–403 (electronic).

[Hir64]   Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326.

[Hir77]   ———, *Idealistic exponents of singularity*, Algebraic geometry (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, MD., 1976), Johns Hopkins Univ. Press, Baltimore, Md., 1977, pp. 52–125.

[Hir83]   ———, *On Nash blowing-up*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Mass., 1983, pp. 103–111.

[HLOQ00]   H. Hauser, J. Lipman, F. Oort, and A. Quirós (eds.), *Resolution of singularities*, Progress in Math., vol. 181, Birkhäuser Verlag, Basel, 2000.

[Jel87]   Zbigniew Jelonek, *The extension of regular and rational embeddings*, Math. Ann. **277** (1987), no. 1, 113–120.

[Kal91]   Shulim Kaliman, *Extensions of isomorphisms between affine algebraic subvarieties of k^n to automorphisms of k^n*, Proc. Amer. Math. Soc. **113** (1991), no. 2, 325–334.

[Kaw06]   Hiraku Kawanoue, *Toward resolution of singularities over a field of positive characteristic Part I.*, arXiv:math.AG/0607009, 2006.

[Kol96]   János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 32, Springer-Verlag, Berlin, 1996.

[Kol97]   ———, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.

[Laz04]   Robert Lazarsfeld, *Positivity in algebraic geometry. I-II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 48, Springer-Verlag, Berlin, 2004.

[Lip75]   Joseph Lipman, *Introduction to resolution of singularities*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 187–230.

[Mur67]   J. P. Murre, *Lectures on an introduction to Grothendieck's theory of the fundamental group*, Tata Institute of Fundamental Research, Bombay, 1967.

[Nar83]   R. Narasimhan, *Monomial equimultiple curves in positive characteristic*, Proc. Amer. Math. Soc. **89** (1983), no. 3, 402–406.

[Par99]   Kapil H. Paranjape, *The Bogomolov-Pantev resolution, an expository account*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., vol. 264, Cambridge Univ. Press, Cambridge, 1999, pp. 347–358.

[Sha94]   Igor R. Shafarevich, *Basic algebraic geometry. 1–2*, Springer-Verlag, Berlin, 1994.

[Vil89]   Orlando Villamayor, *Constructiveness of Hironaka's resolution*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32.

[Vil92]   ———, *Patching local uniformizations*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 6, 629–677.

[Vil96]   ———, *Introduction to the algorithm of resolution*, Algebraic geometry and singularities (La Rábida, 1991), Progr. Math., vol. 134, Birkhäuser, Basel, 1996, pp. 123–154.

[Vil06] ⸻, *Rees algebras on smooth schemes: integral closure and higher differential operators*, arXiv:math.AC/0606795, 2006.

[Wło05] Jarosław Włodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc. **18** (2005), no. 4, 779–822 (electronic).

[Zar39] Oscar Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) **40** (1939), 639–689.

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